PARABOLIC VECTOR BUNDLES AND EQUIVARIANT VECTOR BUNDLES

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Abstract. Given a complex manifold X, a normal crossing divisor $D \subset X$ whose irreducible components $D_1, \ldots, D_s$ are smooth, and a choice of natural numbers $\underline{r} = (r_1, \ldots, r_s)$, we construct a manifold $X(D, \underline{r})$ with an action of a torus Γ and we prove that some full subcategory of the category of Γ-equivariant vector bundles on $X(D, \underline{r})$ is equivalent to the category of parabolic vector bundles on (X, D) in which the lengths of the filtrations over each irreducible component of X are given by $\underline{r}$. When X is Kaehler, we study the Kaehler cone of $X(D, \underline{r})$ and the relation between the corresponding notions of slope-stability.

1. Introduction

1.1. Let X be a (not necessarily compact) complex manifold and let $D \subset X$ be a divisor with normal crossings, whose irreducible components $D_1, \ldots, D_s$ are smooth. Let $\mathscr{P}(X, D, \underline{r})$ be the category of parabolic vector bundles over (X, D) in which the lengths of the filtrations over the irreducible components $D_1, \ldots, D_s$ of D are given by the natural numbers $\underline{r} = (r_1, \ldots, r_s)$ (see Section 8 for a precise definition).

In this paper we construct a manifold $X(D, \underline{r})$ endowed with an action of an algebraic torus Γ and an invariant projection

$$\Pi : X(D, \underline{r}) \to X,$$

we define a full subcategory $\mathscr{V}_\Gamma(X(D, \underline{r}))$ of the category of equivariant vector bundles over $X(D, \underline{r})$, and we construct a functor

$$M : \mathscr{V}_\Gamma(X(D, \underline{r})) \to \mathscr{P}(X, D, \underline{r}),$$

which induces an equivalence of categories (see Theorem 8.1).

When X is Kaehler we describe the Kaehler cone of $X(D, \underline{r})$ in terms of that of X. Afterwards, for any choice of Kaehler class ω of X and parabolic weights Λ we construct a family of Kaehler classes $\Omega(\omega, \Lambda, \epsilon)$ on $X(D, \underline{r})$ parametrized by $\epsilon \in \mathbb{R}_{>0}$ and we prove that for small enough ϵ the notions of (ω, Λ)-parabolic slope stability and that of $\Omega(\omega, \Lambda, \epsilon)$-slope stability for equivariant vector bundles correspond each other by the equivalence of categories (see Theorem 9.3). For any set of weights Λ we have

$$\lim_{\epsilon \to 0} \Omega(\omega, \Lambda, \epsilon) = \Pi^* \omega$$

Date: January 31, 2001.
1991 *Mathematics Subject Classification.* Primary: 14D20; Secondary: 14F05.

so when $\epsilon \to 0$ the volume of the fibres of Π tends to 0. In this sense, our result could be thought of as a statement on stability of vector bundles in the adiabatic limit.

The functor M gives also an equivalence of categories of families of (parabolic, equivariant) vector bundles parametrized by complex spaces S, and in some particular cases our results allow to identify the two moduli problems.

The construction of $X(D,\underline{r}) \to X$ is functorial with respect to X and D in the following sense: if $f : Y \to X$ is a map of complex manifolds which is transverse to D, then we have an induced map $f_{D,\underline{r}}$ which makes the following diagram commutative:

$$\begin{tikzcd} Y(f^{-1}D,\underline{r}) \arrow[r, "f_{D,\underline{r}}"] \arrow[d] & X(D,\underline{r}) \arrow[d] \\ Y \arrow[r, "f"] & X. \end{tikzcd}$$

Furthermore the functor M is compatible with the maps $f_{D,\underline{r}}$, in the sense that the following diagram is also commutative:

$$\begin{tikzcd} \mathscr{V}_\Gamma(X(D,\underline{r})) \arrow[r, "f^*_{D,\underline{r}}"] \arrow[d, "M"'] & \mathscr{V}_\Gamma(Y(f^{-1}D,\underline{r})) \arrow[d, "M"] \\ \mathscr{P}(X,D,\underline{r}) \arrow[r, "f^*"] & \mathscr{P}(Y,f^{-1}D,\underline{r}). \end{tikzcd}$$

The category $\mathscr{V}_\Gamma(X(D,\underline{r}))$ consists of equivariant vector bundles $W \to X(D,\underline{r})$ whose weights over the fixed point set of the action of Γ on $X(D,\underline{r})$ satisfy certain restrictions. Hence, the condition of some equivariant vector bundle W being an object of $\mathscr{V}_\Gamma(X(D,\underline{r}))$ is purely topological.

1.2. The ideas in this paper are very similar of those of Biswas in [B] (see also the references therein). Biswas constructs a finite Galois covering Y of X and shows how to obtain, out of a parabolic bundle over (X,D), a bundle on Y which is equivariant w.r.t. the Galois group of the covering (this establishes a link between the theory of parabolic bundles and the theory of bundles over orbifolds — note that some particular cases of this link were already known before the work of Biswas). Biswas also relates the stability conditions of the parabolic sheaf and of the equivariant one. In contrast with our case, however, the manifold Y depends not only on X, D and $\underline{r}$, but also on the choice of the parabolic weights. It would be interesting to relate the approach of Biswas to that of this paper.

From another point of view, the results which we present here are related to those of García–Prada [GP1, GP2], in which holomorphic pairs are studied in terms of equivariant vector bundles. The ideas of García–Prada have been successfully applied to other situations dealing with holomorphic bundles with extra structure (see for example [A, AGP, BDGW, BGM]).

1.3. The construction of $X(D, \underline{r})$ is made in several steps. First we consider the case of D smooth and $r = r_1 = 1$. We define $X(D, 1)$ (or X_D for short) as a family of conics over X which degenerate precisely over D. This is the same thing as the blow up of $X \times \mathbb{P}^1$ along $D \times \{[0 : 1]\}$, which is (a compactification of) the deformation to the normal cone of D. The manifold X_D inherits an action from the diagonal action of $\mathbb{C}^*$ on $X \times \mathbb{P}^1$ which is trivial on the X factor and which is defined on $\mathbb{P}^1$ as $\theta \cdot [y : w] := [y : \theta w]$. Next we consider the case of D smooth and $r > 1$. We construct a tower of manifolds

$$X(D, r) = Y_r \to Y_{r-1} \to \cdots \to Y_0 = X$$

by applying recursively the previous construction, so that $Y_{j+1} = (Y_j)_{D_j}$. Here $D_j \subset Y_j$ is a certain smooth divisor which lies over D, and each manifold Y_j carries an action of $(\mathbb{C}^\times)^j$.

Finally, when $D = D_1 \cup \cdots \cup D_s$ is a normal crossing divisor, we define $X(D, \underline{r})$ to be the fibred product

$$\Pi : X(D, \underline{r}) = X(D_1, r_1) \times_X \cdots \times_X X(D_s, r_s) \to X.$$

This carries a diagonal action of $\Gamma = G(r_1) \times \cdots \times G(r_s)$. The fibres of Π over $X \setminus D$ are products of $\mathbb{P}^1$'s, and those over D are singular, their irreducible components being products of $\mathbb{P}^1$'s. In fact, $X(D, \underline{r})$ can be constructed by making a sequence of blow ups along subvarieties of a product $X \times \mathbb{P}^1 \times \ldots \mathbb{P}^1$ which lie above D.

The category $\mathscr{V}_\Gamma(X(D, \underline{r}))$ and the functor M are also defined inductively, following the construction of $X(D, \underline{r})$. The main point is to define M in the case D smooth and $r = 1$, since everything is built up by applying recursively the construction in this simplest case. The proof that M induces an equivalence of categories and some of the computations needed to prove the relation with Mumford–Takemoto stability are also obtained by reducing to this simple case.

1.4. We now explain the contents of the following sections. In Section 2 we give some definitions and results on equivariant vector bundles which will be used along the paper. In Section 3 we describe the construction of the $\mathbb{C}^\times$-manifold X_D out of the pair (X, D); as we said before, we will obtain the manifold $X(D, \underline{r})$ by iterating this construction. In Section 4 we define a functor from (some full subcategory of) the category of $\mathbb{C}^\times$-bundles over X_D to that of parabolic vector bundles over (X, D) with $r = r_1 = 1$, and we prove that it induces an equivalence of categories. In Section 5 we extend the result of the previous section to arbitrary parabolic bundles over a smooth divisor. Section 6 is devoted to studying the Kaehler cone of the manifolds $X(D, r)$ and some topological aspects of the equivalence between equivariant and parabolic vector bundles. These results are used in Section 7 to relate the notions of slope and parabolic slope of vector bundles. In Section 8 everything is extended to the case of a normal crossing divisor. In Section 9 we study the stability condition. Finally, in Section 10 we consider the case of X being a Riemann surface.

1.5. **Notations and conventions.** Unless we say the contrary, the following will be implicitly assumed in this paper: all vector bundles will be complex, all metrics on vector bundles will be Hermitian, all vector bundles, maps of vector bundles, manifolds,

and actions of groups on manifolds will be holomorphic. A divisor will mean a reduced divisor.

We will use the following notations. If Y is a complex manifold $\mathrm{Bihol}(Y)$ will denote the group of biholomorphisms of Y with itself. If G is an abelian group, Y is a G manifold, $V \to Y$ is a G-equivariant vector bundle and $Y' \subset Y^G$, then $\chi_G(V|_{Y'})$ will denote the set of characters appearing in the decomposition of $V|_{Y'}$ in bundles of irreps of G (as usual the superscript ^{G} denotes the fixed points). We will call $\chi_G(V|_{Y'})$ the set of G-weights of V on Y'.

If $\mathcal{C}$ is any category, we will usually write $A \in \mathcal{C}$ to mean that A is an object of $\mathcal{C}$.

1.6. **Aknowledgements.** Most of this paper was done during a stay at the Centre de Mathématiques of the École Polytechnique. The author is very pleased to have the opportunity to thank this institution for financial support and for providing excellent working conditions. Thanks are also due to C. Sabbah, V. Navarro Aznar for some useful comments.

2. Equivariant bundles

2.1. **Weights of bundles.** We will say that an inclusion $Y \subset Z$ of topological spaces is a strong inclusion if every connected component of Z contains some point of Y. Assume that a group G acts on a vector bundle $W \to Z$ linearly on the fibres. If $Z' \subset Z^G$, and $Y \subset Z'$ is a strong inclusion, then

$$\chi_G(W|_Y) = \chi_G(W|_Z).$$

This follows from the local invariance of the set of weights on the fixed point locus.

2.2. **Stable and unstable sets.** Let Y be a manifold with an action of $\mathbb{C}^\times$. Denote the image of $\theta \in \mathbb{C}^\times$ acting on $y \in Y$ by $\theta \cdot y$.

Let Y' be a connected component of $Y^{\mathbb{C}^\times}$. Define the stable (resp. unstable) set $U^+(Y')$ (resp. $U^-(Y')$) of Y' to be

$$U^+(Y') = \{y \in Y \mid \lim_{\mathbb{C}^\times \ni \theta \to 0} \theta \cdot y \text{ exists and belongs to } Y'\},$$
$$U^-(Y') = \{y \in Y \mid \lim_{\mathbb{C}^\times \ni \theta \to \infty} \theta \cdot y \text{ exists and belongs to } Y'\}.$$

One can prove that the sets $U^+(Y')$ and $U^-(Y')$ are smooth complex submanifolds of Y (this is part of the Kaehler version of Białynicki-Birula theorem, see [CS]). In the cases which will be considered in this paper this fact will be obvious.

Let $W \to Y$ be a complex bundle with a linear action of $\mathbb{C}^\times$ lifting the one on Y. We will denote the fibre over y by W_y. For any fixed point $y \in Y^{\mathbb{C}^\times}$ the fibre W_y gets a linear action of $\mathbb{C}^\times$. The isomorphism class of this representation depends only on the connected component Y' of $Y^{\mathbb{C}^\times}$ to which y belongs. Let $W_y = \bigoplus_{k \in \mathbb{Z}} W_y(k)$ be the decomposition of W_y in weights.

Lemma 2.1. *Consider the standard inclusion of groups $S^1 \subset \mathbb{C}^\times$, and take on Y the action of S^1 induced by restriction. Let $y \in Y^{S^1}$ be a fixed point. There exists a S^1 invariant neighbourhood $A \subset Y$ of y and a holomorphic and S^1-equivariant trivialisation $\phi : W|_A \xrightarrow{\simeq} A \times W_y$, with $A \times W_y$ supporting the diagonal action of S^1. (We will say that such a trivialisation is centered around y.)*

Proof. Let A_0 be a S^1-invariant neighbourhood of y, small enough so that there is a holomorphic trivialisation of $W|_{A_0}$. Then the map $e : \Gamma := \Gamma(A_0; W) \to W_y$ given by evaluation at y (where $\Gamma(A_0; W)$ denotes the set of holomorphic sections defined on A_0) is exhaustive. The group S^1 acts on Γ by pullback. Let $\Gamma = \bigoplus_{k\in\mathbb{Z}} \Gamma(k)$ be the splitting given by the weights. Since e is exhaustive, there exists elements $w_1, \dots, w_N$ (where $N = \operatorname{rk} W$) and weights $k_1, \dots, k_N$ such that $w_j \in \Gamma(k_j)$ and such that $e(w_1), \dots, e(w_N)$ span W_y (and so form a basis). Consequently, there is a neighbourhood $A \subset A_0$ of y over which the sections $\{w_j\}$ trivialise W. □

Remark 2.2. *Note that $Y^{S^1} = Y^{\mathbb{C}^\times}$. The obtained trivialisation is weakly $\mathbb{C}^\times$-equivariant in the following sense. If $y \in A$, $\theta \in \mathbb{C}^\times$ and θ can be joined to $1 \in \mathbb{C}^\times$ by a path $\{\theta(t), 0 \le t \le 1\}$ so that $\theta(t) \cdot y \in A$ for all $0 \le t \le 1$, then for any $v \in W_y$ we have $\phi(\theta \cdot v) = \theta \cdot \phi(v)$. This follows from the fact that both the trivialisation ϕ and the action of $\mathbb{C}^\times$ on W are holomorphic.*

Take a metric on W. Chose a connected component Y' of the fixed point set and define, for any $r \in \mathbb{R}$,

$$W^{+,r}(Y') = \{w \in W|_{U^+(Y')} \mid \lim_{\theta\to 0} |\theta^{-r}(\theta \cdot w)| < \infty\}.$$

We call $W^{+,r}(Y')$ the r-stable subbundle of W towards Y'.

Lemma 2.3. *The family of sets $\{W^{+,r}(Y') \mid r \in \mathbb{R}\}$ is a decreasing filtration of $\mathbb{C}^\times$-invariant subbundles of $W|_{U^+(Y')}$, and it is independent of the chosen metric on W. Let $\phi : W \to W'$ be an equivariant map of $\mathbb{C}^\times$-bundles. Then, for any $r \in \mathbb{R}$, we have $\phi(W^{+,r}|_{U^+(Y')}) \subset W'^{+,r}|_{U^+(Y')}$.*

Proof. Follows from the definitions and Lemma 2.1 together with Remark 2.2. □

Suppose that the only weight of W in Y' is zero. For any $z \in U^-(Y')$ define the map $\rho_W^z : W_z \to W_{z_-}$ by $\rho_W^z(w) := \lim_{\theta\to\infty} \theta \cdot w$. Similarly, if $z \in U^+(Y')$, we set $R_W^z : W_z \to W_{z_+}$ to be $R_W^z(w) := \lim_{\theta\to 0} \theta \cdot w$.

Lemma 2.4. *If $z \in U^-(Y')$ (resp. $z \in U^+(Y')$) then ρ_W^z (resp. R_W^z) is well defined and is an isomorphism of vector spaces.*

Proof. This also follows from Lemma 2.1 and Remark 2.2. □

2.3. **Closure of subbundles.** Let Z be a manifold, let $Y = Z \times \mathbb{C}$, and consider the action of $\mathbb{C}^\times$ on Y given by $\theta \cdot (z, a) = (z, \theta a)$. The fixed point set of this action is $Y' = Z \times \{0\}$, and we have $U^+(Y') = Y$. Define $Y^* = Y \setminus Y'$. Let $p : Y \to Z$ be the projection.

Lemma 2.5. *Let $W \to Y$ be a $\mathbb{C}^\times$-vector bundle whose only weight in Y' is 0. Let $W' \subset W|_{Y^*}$ be a $\mathbb{C}^\times$-invariant subbundle. Then there is a unique $\mathbb{C}^\times$-invariant subbundle $\overline{W'} \subset W$ such that $\overline{W'}|_{Y^*} = W'$. Furthermore, if $V \to Y$ is another $\mathbb{C}^\times$-vector bundle whose only weight at Y' is 0 and $\phi : W \to V$ is a $\mathbb{C}^\times$-equivariant map such that $\phi|_{W'} : W' \to V|_{Y^*}$ is exhaustive, then $\phi|_{\overline{W'}} : \overline{W'} \to V$ is also exhaustive.*

Proof. We first prove uniqueness. Let $\overline{W'} \subset W$ be such an extension. Necessarily, the only weight of $\overline{W'}$ in Y' is 0. Let $z_0 \in Z$ be any point and define $z = (z_0, 1) \in Y$, so that $z_+ = (z_0, 0)$. By Lemma 2.4, $R^z_{\overline{W'}}$ is an isomorphism and by equivariance we have $R^z_{\overline{W'}} = R^z_W|_{W'_z}$. Hence we must have $\overline{W'}_{(z_0,0)} = R^z_W(W'_z) \subset W_{(z_0,0)}$. In other words, the fibre of $\overline{W'}$ over $(z_0, 0)$ is determined by R^z_W and by the fibre W'_z. The last claim of the lemma follows also easily using the isomorphisms R^z.

To prove existence it suffices to work locally (thanks to uniqueness). So take any $z_0 \in Z$. By Lemma 2.1 there exists a neighbourhood $A \subset Y$ of $(z_0, 0)$ and a weakly $\mathbb{C}^\times$-equivariant trivialisation

$$\phi : W|_A \to A \times W_{(z_0,0)},$$

where $\mathbb{C}^\times$ acts trivially on $W_{(z_0,0)}$. By shrinking A if necessary, we may assume that

$$A = \{(b, a) \in Z \times \mathbb{C} \mid b \in p(S),\ |a| < \epsilon\}$$

for some $\epsilon > 0$. Let $A^* = A \cap Y^*$. Using the trivialisation ϕ the subbundle $W'|_{S^*}$ is described by a map to the Grassmannian of the fibre over $(z_0, 0)$:

$$\Psi : A^* \to \mathrm{Gr}(W_{(z_0,0)}).$$

Since W' is $\mathbb{C}^\times$-invariant and holomorphic, we deduce that Ψ is weakly $\mathbb{C}^\times$-equivariant and holomorphic, i.e., $\Psi(z, a) = \Psi(z, b)$ whenever $0 < |a| < \epsilon$ and $0 < |b| < \epsilon$. It then follows that we can extend Ψ to a map $\overline{\Psi} : A \to \mathrm{Gr}(W_{(z_0,0)})$ by simply setting $\overline{\Psi}(z, a) = \Psi(z, \epsilon/2)$. This map describes the desired weakly $\mathbb{C}^\times$-equivariant subbundle of $W|_A$. ☐

The preceding lemma holds also true when W has a unique weight on Y' (not necessarily zero). However, if W has more than one weight in Y' then the result might be false. Indeed, consider the trivial bundle $\underline{\mathbb{C}^2} \to \mathbb{C}^2$ (i.e., here $Z = \mathbb{C}$). Let $\langle e_1, e_2 \rangle$ be the canonical basis of the fibre $\mathbb{C}^2$. Let $L \subset \underline{\mathbb{C}^2}$ be the line bundle defined over $\mathbb{C} \times \mathbb{C}^\times$ whose fibre over (z, a) is $L_{(z,a)} = \langle z e_1 + a e_2 \rangle$. Consider the action of $\mathbb{C}^\times$ on $\mathbb{C}^2$ given by $\theta \cdot (z, a) = (z, \theta a)$ and its lift to the trivial bundle $\underline{\mathbb{C}^2}$ defined by $\theta \cdot (\alpha e_1 + \beta e_2) = \alpha e_1 + \theta \beta e_2$. Then L is $\mathbb{C}^\times$-invariant, but it does not extend to a line subbundle of $\underline{\mathbb{C}^2} \to \mathbb{C}^2$.

Corollary 2.6. *Let $W_i \to Y$, $i = 0, 1$, be $\mathbb{C}^\times$-equivariant bundles whose only weight in Y' is 0, and let $\phi : W_0|_{Y^*} \to W_1|_{Y^*}$ be a $\mathbb{C}^\times$-equivariant map. Then ϕ extends to a $\mathbb{C}^\times$-equivariant map from W_0 to W_1.*

Proof. Let $W = W_0 \oplus W_1$ and let W' be the graph of ϕ. The only weight of W in Y' is 0, and W' is a $\mathbb{C}^\times$-equivariant subbundle of $W|_{Y^*}$. Hence by Lemma 2.5 W' extends to a $\mathbb{C}^\times$-equivariant subbundle $\overline{W'} \subset W$. To see that $\overline{W'}$ is the graph of a map from W_0 to W_1 it suffices to check that the projection $\overline{W'} \to W_0$ is an isomorphism, and this follows from the last claim of Lemma 2.5. □

2.4. **Lifting actions to line bundles.** Let X be a manifold, and let $D \subset X$ be a smooth divisor. Let $\pi : L \to X$ be a line bundle and $\sigma \in H^0(X; L)$ a section which is transverse to the zero section and such that $\sigma^{-1}(0) = D$.

Lemma/Definition 2.7. *Let* $\mathrm{Bihol}(X, D)$ *be the group of biholomorphisms of* X *which preserve* D*, and let* $\mathrm{Bihol}(L)$ *be the group of biholomorphisms of* L *which preserve* $\pi^{-1}(D)$ *and which map fibres linearly to fibres. There exists a right inverse* $\phi_\sigma : \mathrm{Bihol}(X, D) \to \mathrm{Bihol}(L)$ *to the projection map* $\mathrm{Bihol}(L) \to \mathrm{Bihol}(X, D)$.

Proof. Let $f \in \mathrm{Bihol}(X, D)$. We first define the restriction f_0 of $\phi_\sigma(f)$ to $L_0 := L \backslash \pi^{-1}(D)$ as follows: for any $x \in L_0$ we set

$$f_0(x) = \frac{x}{\sigma(\pi(x))} \sigma(f\pi(x)) \in L_{f\pi(x)}$$

(note that the fraction in the RHS is a complex number). The map f_0 can be equivalently seen as a map of line bundles

$$F : L|_{X \backslash D} \to f^* L|_{X \backslash D},$$

and to see that f_0 extends to a map from L to L it suffices to prove that F extends to a map of line bundles defined over the whole X. For that, and thanks to Riemann's extension theorem, it is enough to check that if we fix a metric on L and $K \subset X$ is a compact subset, then the restriction of F to $K \cap (X \setminus D)$ is bounded. This is the same thing as

$$\sup \left\{ \frac{|\sigma(fy)|}{|\sigma(y)|} \ : \ y \in K \cap (X \setminus D) \right\} < \infty,$$

and this follows from the fact that f is a biholomorphism which preserves D and that σ is transverse to the zero section. Hence f_0 extends to a map $\phi_\sigma(f) : L \to L$ which lifts f. It is also clear that $\phi_\sigma(\mathrm{Id}) = \mathrm{Id}$, and that if $f, g \in \mathrm{Bihol}(X, D)$ then $\phi(s)(fg) = \phi_\sigma(f)\phi_\sigma(g)$ (indeed, that $f_0 g_0 = (fg)_0$ is obvious from the definition, and then use that $X \setminus D$ is dense in X). Finally, it follows from the lattest property that if $f \in \mathrm{Bihol}(X, D)$ then $\phi_\sigma(f)\phi_\sigma(f^{-1}) = \mathrm{Id}$, so $\phi_\sigma(f)$ is indeed a biholomorphism. (Note that $\phi_\sigma(f)$ can be characterized as the unique element of $\mathrm{Bihol}(L)$ which lifts f and which makes σ equivariant.) □

Lemma 2.8. *If* $f \in \mathrm{Bihol}(X, D)$*, then the restriction of* $\phi_\sigma(f)$ *to* $\pi^{-1}(D)$ *can be computed as follows. Let* $z \in D$*. Since* σ *is transverse to zero,* $d\sigma(z) : N_z \to L_z$ *is an isomorphism, where* $N \to D$ *is the normal bundle of* $D \subset X$*. On the other hand,* $df : N_z \to N_{f(z)}$ *is also an isomorphism and, furthermore, that*

$$\phi_\sigma(f)|_{L_z} = d\sigma(f(z)) \circ df \circ d\sigma(z) : L_z \to L_{f(z)}.$$

Proof. It follows from a computation in local coordinates. □

Lemma 2.9. *If a group Γ acts on X by biholomorphisms preserving D, then there is a unique linear lift of the action of Γ to L for which σ is Γ-equivariant. If $x \in X^{\Gamma}$, the character χ of Γ acting on L_x is 1 if $\sigma(x) \neq 0$, and if $\sigma(x) = 0$ then χ is equal to the character of the action of Γ on the fibre N_x of the normal bundle $N \to \sigma^{-1}(0)$.*

Proof. The first part of the lemma follows from Lemma 2.7, and the second part from Lemma 2.8. □

3. The $\mathbb{C}^{\times}$-manifold X_D

Let X be a smooth manifold (not necessarily compact), and let $D \subset X$ be a smooth divisor. Let $L \to X$ be a line bundle and $\sigma \in H^0(L)$ a section which is transverse to zero and such that $\sigma^{-1}(0) = D$. Let

$$X_{D,\sigma} := \{[x : y : w] \in \mathbb{P}(L \oplus \underline{\mathbb{C}} \oplus \underline{\mathbb{C}})_z \mid z \in X,\ xy = w^2\sigma(z)\},$$

and define an action of $\mathbb{C}^{\times}$ on X_D by $\theta \cdot [x : y : w] := [\theta^2 x : y : \theta w]$ for any $\theta \in \mathbb{C}^{\times}$. If $\sigma' \in H^0(L)$ is another nonzero section which is transverse to zero then $\sigma' = \theta\sigma$ for some $\theta : X \to \mathbb{C}^{\times}$, so the map $\Theta : X_{D,\sigma} \to X_{D,\sigma'}$ which sends $[x : y : w]$ to $[\theta x : y : w]$ is a biholomorphism. In view of this, we will just write, to save on notation, X_D instead of $X_{D,\sigma}$. (But it is important to keep in mind that whereas the assignement $(X, D, \sigma) \to X_{D,\sigma}$ is functorial, this is not the case of the assignement $(X, D) \to X_D$.) We will denote by $p : X_D \to X$ the projection.

It follows from an easy local computation that X_D is smooth.

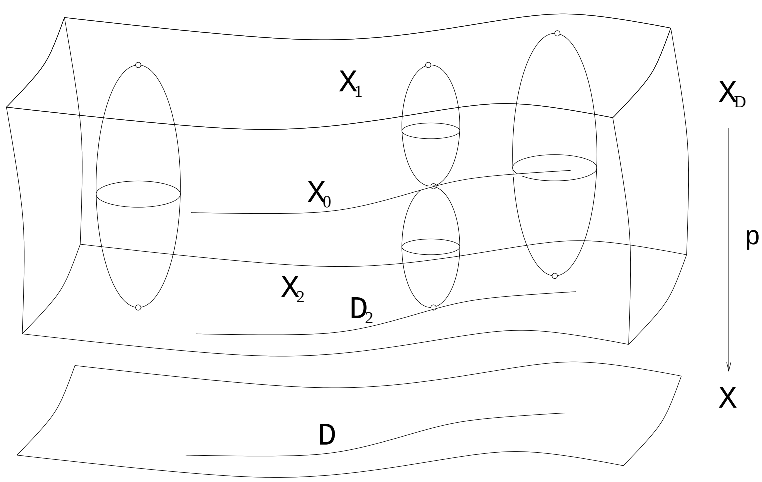

Figure 1. The manifold X_D. The spheres represent the closure of orbits of the action of $\mathbb{C}^{\times}$.

Lemma/Definition 3.1. *There exists a morphism of groups ϕ from* $\mathrm{Bihol}(X, D)$ *to* $\mathrm{Bihol}(X_D)$ *such that for any $f \in \mathrm{Bihol}(X, D)$ we have $p \circ \phi(f) = f \circ p$.*

Proof. Use Lemma 2.7 to get a lift $\phi_\sigma(f) : L \to L$ of f. Taking the trivial lift of the action to the trivial bundle $\underline{\mathbb{C}}$, this gives an element of $\mathrm{Bihol}(\mathbb{P}(L \oplus \underline{\mathbb{C}} \oplus \underline{\mathbb{C}}))$ which leaves X_D fixed, and we define $\phi(f)$ to be the induced element of $\mathrm{Bihol}(X_D)$. $\square$

In the sequel we will denote, for any $f \in \mathrm{Bihol}(X, D)$, $f_D := \phi(f)$.

The fixed point set of the action of $\mathbb{C}^\times$ on X_D is the disjoint union of the following submanifolds:

$$X_1 = \{[x : y : w] = [0 : 1 : 0]\}, \qquad X_2 = \{[x : y : w] = [1 : 0 : 0]\},$$
$$X_0 = \{[x : y : w] = [0 : 0 : 1]\} \cap X_D.$$

There are canonical identifications given by the projection p:

$$X_2 \simeq X \simeq X_1 \qquad \text{and} \qquad X_0 \simeq D.$$

Using this identifications we have, for any $f \in \mathrm{Bihol}(X, D)$,

$$f_D|_{X_1} = f_D|_{X_2} = f \qquad \text{and} \qquad f_D|_{X_0} = f|_D.$$

Denote by $D_2 \subset X_D$ the copy of D obtained by means of the identification $X_2 \simeq X$.

Define $\Delta^+ = \Delta^+(X, D) := \overline{U^+(X_0)}$ and $\Delta^- = \Delta^-(X, D) := \overline{U^-(X_0)}$. Both Δ^+ and Δ^- are smooth $\mathbb{C}^\times$-invariant divisors in X_D, and $\Delta^+ \cap X_2 = D_2$. We can explicitly describe them as

$$\Delta^+ = \{y = 0\} \cap X_D \qquad \text{and} \qquad \Delta^- = \{x = 0\} \cap X_D. \tag{3.1}$$

Let us denote by $N^+ \to \Delta^+$ (resp. $N^- \to \Delta^-$) the normal bundle of the inclusion $\Delta^+ \subset X_D$ (resp. $\Delta^- \subset X_D$).

Lemma 3.2. *The fixed point set $(\Delta^+)^{\mathbb{C}^\times}$ is the disjoint union of $X_0 = \Delta^+ \cap X_0$ and $X_2' := \Delta^+ \cap X_2$. The weight of the action of $\mathbb{C}^\times$ on the restriction of N^+ to X_0 (resp. X_2') is -1 (resp. 0).*

Proof. The first statement is obvious. To prove the statement on the weights, observe that since Δ^+ and X_2 intersect transversely, $N^+|_{X_2'}$ can be identified (in a $\mathbb{C}^\times$-equivariant way) with the normal bundle of the inclusion $X_2' \subset X_2$. But since the action of $\mathbb{C}^\times$ on X_2 is trivial, it follows that the weight of $\mathbb{C}^\times$ acting on $N^+|_{X_2'}$ is 0. On the other hand, $N^+|_{X_0}$ can be identified with the normal bundle of the inclusion $X_0 \subset U^-(X_0)$. Now, since $U^-(X_0) = \{[0 : y : 1]\} \cap X_D$, and $\theta \cdot [0 : y : 1] = [0 : y : \theta] = [0 : \theta^{-1}y : 1]$, it follows that the action of $\mathbb{C}^\times$ has weight -1. $\square$

Lemma 3.3. *The fixed point set $(\Delta^-)^{\mathbb{C}^\times}$ is the disjoint union of $X_0 = \Delta^- \cap X_0$ and $X_1' := \Delta^+ \cap X_1$. The weight of the action of $\mathbb{C}^\times$ on the restriction of N^- to X_0 (resp. X_1') is 1 (resp. 0).*

Proof. Exactly like that of the preceding lemma, but using that $N^-|_{X_0}$ can be identified with the normal bundle of the inclusion $X_0 \subset U^+(X_0)$, and that $U^+(X_0) = \{[x : 0 : 1]\} \cap X_D$ and $\theta \cdot [x : 0 : 1] = [\theta^2 x : 0 : \theta] = [\theta x : 0 : 1]$. □

3.1. **Example: The manifold H_n.** Define for any natural number $n \geq 1$

$$H_n = \{(t_1, \dots, t_n, [x : y : w]) \mid xy = w^2 t_1\} \subset \mathbb{C}^n \times \mathbb{P}(\mathbb{C}^3).$$

Consider the action of $\mathbb{C}^\times$ on H_n given by $\theta \cdot (t_1, \dots, t_n, [x : y : w]) = (t_1, \dots, t_n, [\theta^2 x : y : \theta w])$. Let $p_n : H_n \to \mathbb{C}^n$ be the projection. Note that the $\mathbb{C}^\times$-manifold H_n is nothing but $\mathbb{C}^n_{\{0\}\times\mathbb{C}^{n-1}}$.

For future use we define the map $\xi : H_n \to \mathbb{R}$ as follows

$$\xi(t_1, \dots, t_n, [x : y : w]) := -\frac{2|x|^2 + |w|^2}{|x|^2 + |y|^2 + |w|^2}.$$

Up to some multiplicative constant, this is the moment map of the action of $S^1 \subset \mathbb{C}^\times$ on H_n with respect to the symplectic structure inherited by the inclusion $H_n \subset \mathbb{C}^n \times \mathbb{P}(\mathbb{C}^3)$ and the usual symplectic structures on $\mathbb{C}^n$ and $\mathbb{P}(\mathbb{C}^3)$. Finally, we define the following subset of H_n:

$$U_n = \{(t_1, \dots, t_n, [x : y : w]) \in H_n \mid x \neq 0\} = U^+(\{[x : y : w] = [1 : 0 : 0]\}).$$

4. An equivalence of categories

4.1. **Definitions of $\mathscr{V}_{\mathbb{C}^\times}(X_D)$, $\mathscr{P}(X, D)$ and μ.** We define the category $\mathscr{V}_{\mathbb{C}^\times}(X_D)$ as follows:

1. The objects of $\mathscr{V}_{\mathbb{C}^\times}(X_D)$ are $\mathbb{C}^\times$-equivariant vector bundles $W \to X_D$ such that the only weight in $W|_{X_1 \cup X_2}$ is 0 and the weights in $W|_{X_0}$ belong to $\{0, 1\}$.
2. The morphisms between two $\mathbb{C}^\times$-equivariant bundles W, W' are the $\mathbb{C}^\times$-equivariant maps of vector bundles $\psi : W \to W'$.

We define the category $\mathscr{P}(X, D)$ as follows:

1. The objects of $\mathscr{P}(X, D)$ are pairs (V, V_1) consisting of a vector bundle $V \to X$ together with a subbundle $V_1 \subset V|_D$.
2. The morphisms between two objects (V, V_1) and (V', V_1') are the maps of vector bundles $\phi : V \to V'$ such that $\phi|_D(V_1) \subset V_1'$.

4.1.1. *The functor μ.* We are now going to define a functor $\mu : \mathscr{V}_{\mathbb{C}^\times}(X_D) \to \mathscr{P}(X, D)$. By a slight abuse of notation, in this subsection we will write $D = D_2$ and $X = X_2$.

We first define μ acting on objects. Let $W \to X_D$ be a $\mathbb{C}^\times$-bundle. Let $W^+ := W^{+,1}(X_0) \to U^+(X_0)$ be the 1-stable subbundle of W towards X_0 (see Subsection 2.2). The bundle W^+ extends to a unique $\mathbb{C}^\times$-invariant subbundle $\overline{W^+} \subset W|_{\Delta^+}$. Indeed, locally around any $z \in D \subset X_D$, X_D is $\mathbb{C}^\times$-equivariantly biholomorphic to $Y = Z \times \mathbb{C}$ with $Z = \mathbb{C}^n$ and the action of $\mathbb{C}^\times$ described in 2.3. Hence, by Lemma 2.5, there exists the closure $\overline{W^+} \subset W$ of W^+ near z. By uniqueness we can make the same reasoning

around all $z \in D$ and patch the resulting local closures of W^+, thus getting the desired extension $\overline{W^+}$. Now, we define

$$\mu(W) := (W|_X, \overline{W^+}|_D).$$

We state the following lemma for later use.

Lemma 4.1. *Let $W \in \mathscr{V}_{\mathbb{C}^\times}(X_D)$, and let $W_0 \subset W|_{\Delta^+}$ be a $\mathbb{C}^\times$-equivariant subbundle, and assume that $\chi_{\mathbb{C}^\times}(W_0|_{X_0}) = \{1\}$. Let $(V, V_1) = \mu(W)$. Then, identifying D with $\Delta^+ \cap X_2$ we have an inclusion of vector bundles over D:*

$$W_0|_{\Delta^+ \cap X_2} \subset V_1.$$

Proof. It follows easily from the definition of μ. □

To define μ acting on morphisms, observe that if $\psi : W \to W'$ is a $\mathbb{C}^\times$-equivariant map of vector bundles then Lemma 2.3 implies that $\psi(W^+) \subset W'^+$. From this it follows easily that $\psi(\overline{W^+}) \subset \overline{W'^+}$. Hence, the restriction of ψ to X is a morphism in the category $\mathscr{P}(X, D)$ between $\mu(W)$ and $\mu(W')$, and we define

$$\mu(\psi) := \psi|_X \in \mathrm{Mor}_{\mathscr{P}(X,D)}(\mu(W), \mu(W')).$$

4.1.2. It is straightforward to deduce from the previous definition that if $W, W', W'' \in \mathscr{V}_{\mathbb{C}^\times}(X_D)$ and $\psi \in \mathrm{Mor}_{\mathscr{V}_{\mathbb{C}^\times}(X_D)}(W, W')$ and $\xi \in \mathrm{Mor}_{\mathscr{V}_{\mathbb{C}^\times}(X_D)}(W', W'')$ then

$$\mu(\xi \circ \psi) = \mu(\xi) \circ \mu(\psi).$$

This proves that μ is indeed a functor.

The following is easily checked: if $f : Y \to X$ is a map whose image is transverse to D (so that $f^{-1}(D) \subset Y$ is a smooth divisor) and $f_D : Y_{f^{-1}(D)} \to X_D$ is the map induced by taking on Y the pullback of L and $\sigma \in H^0(L)$, then we get two commutative diagrams:

$$\begin{tikzcd} Y_{f^{-1}D} \arrow[r, "f_D"] \arrow[d] & X_D \arrow[d] \\ Y \arrow[r, "f"] & X, \end{tikzcd} \qquad \begin{tikzcd} \mathscr{V}_{\mathbb{C}^\times}(X_D) \arrow[r, "f_D^*"] \arrow[d, "\mu"'] & \mathscr{V}_{\mathbb{C}^\times}(Y_{f^{-1}(D)}) \arrow[d, "\mu"] \\ \mathscr{P}(X, D) \arrow[r, "f^*"] & \mathscr{P}(Y, f^{-1}(D)), \end{tikzcd} \tag{4.2}$$

where the horizontal arrows in the RHS are the functors induced by pullbacks. Note that a particular case of such a map $f : Y \to X$ is any biholomorphism $f \in \mathrm{Bihol}(X)$.

Theorem 4.2. *The functor $\mu : \mathscr{V}_{\mathbb{C}^\times}(X_D) \to \mathscr{P}(X, D)$ induces an equivalence of categories.*

Remark 4.3. *It is natural to ask in view of the preceding theorem what happens if we consider $\mathbb{C}^\times$-equivariant bundles on X_D which do not belong to $\mathscr{V}_{\mathbb{C}^\times}(X_D)$. A partial answer to this question is the following: the category consisting of $\mathbb{C}^\times$-equivariant bundles whose weights on $X_D|_{X_1 \cup X_2}$ are 0 and whose weights in $X_D|_{X_0}$ belong to $\{0, k\}$ for some $k \in \mathbb{N}$ (instead of $\{0, 1\}$) is equivalent to the category consisting of pairs (V, V'), where*

$V \to X$ is a bundle and where V' is a subbundle of the restriction of D to the k-th thickening D_k of D (if $\mathscr{I} \subset \mathscr{O}_X$ is the ideal sheaf defining D, then $D_k = \mathrm{Spec}(\mathscr{O}_X/\mathscr{I}^k)$). This can be proved by using the same techniques as here. What seems more difficult is to understand the category consisting of all the $\mathbb{C}^\times$-equivariant bundles on X_D in terms of some category of bundles on X with extra structure (defined somehow along D).

The proof of the theorem is given in Subsections 4.2 and 4.4. The scheme of the proof is the following. First we prove that for any pair of objects $W, W' \in \mathscr{V}_{\mathbb{C}^\times}(X_D)$ the map

$$\mu_{W,W'} : \mathrm{Mor}_{\mathscr{V}_{\mathbb{C}^\times}(X_D)}(W, W') \to \mathrm{Mor}_{\mathscr{P}(X,D)}(\mu(W), \mu(W'))$$

induced by the functor μ is a bijection. In the second part of the proof we will construct, for any $(V, V_1) \in \mathscr{P}(X, D)$, an equivariant bundle $W \in \mathscr{V}_{\mathbb{C}^\times}(X_D)$ such that $\mu(W)$ is isomorphic to (V, V_1) (we use for that the result obtained in the first part). That these two steps suffice to prove that μ induces an equivalence of categories is assured by Freyd's theorem (see Theorem 1.13 in Chapter 1 of [GM]).

Corollary 4.4. *Let $W \to X_D$ be a $\mathbb{C}^\times$-equivariant bundle whose weights in X_0, X_1 and X_2 are zero. Then, using the identification $X \simeq X_2$ to write $p : X_D \to X_2$, we have a canonical isomorphism $W \simeq p^*(W|_{X_2})$.*

Proof. Let W be such a vector bundle. Then $W \in \mathscr{V}_{\mathbb{C}^\times}(X_D)$, and $\mu(W) = (V, 0)$ for some vector bundle $V \to X$. On the other hand, if we define $W_0 := p^*V$ and take on W_0 the trivial action of $\mathbb{C}^\times$, it turns out that $\mu(W_0) = (V, 0)$ as well. Hence, by Theorem 4.2, W and W_0 are isomorphic as $\mathbb{C}^\times$-vector bundles. □

4.2. **The map $\mu_{W,W'}$ is injective.** Let $W, W' \in \mathscr{V}_{\mathbb{C}^\times}(X_D)$ be two $\mathbb{C}^\times$-vector bundles. Let $U = U^-(X_2) = \{z \in X_D \mid z_- \in X_2\}$. Since the only weights of W at X_2 are 0, it turns out that if $z \in U$ then the map $\rho^z_W : W_z \to W_{z_-}$ is an isomorphism (see Lemma 2.4). Similarly, we have an isomophism $\rho^z_{W'} : W'_z \to W_{z_-}$ whenever $z \in U$. Now, a morphism $\phi \in \mathrm{Mor}_{\mathscr{V}_{\mathbb{C}^\times}(X_D)}(W, W')$ is by definition a $\mathbb{C}^\times$-equivariant map $\phi : W \to W'$, and equivariance means that for any $w \in W_z$ and $\theta \in \mathbb{C}^\times$ we have $\phi_{\theta \cdot z}(\theta \cdot w) = \theta \cdot \phi_z(w)$, where $\phi_x : W_x \to W'_x$ is the restriction of ϕ to the fibres over x. Now suppose that $z \in U$ and make $\theta \to \infty$. We get at the limit that $\phi_{z_-} \circ \rho^z_W = \rho^z_{W'} \circ \phi_z$, and since $\rho^z_{W'}$ is an isomorphism we can write

$$\phi_z = (\rho^z_{W'})^{-1} \circ \phi_{z_-} \circ \rho^z_W,$$

which means that $\phi|_U$ is determined by $\phi|_{X_2}$. And since $U \subset X_D$ is dense, it follows that ϕ is also determined by $\phi|_{X_2}$. Hence, $\mu_{W,W'}$ is injective.

4.3. **The map $\mu_{W,W'}$ is exhaustive.** Let $\phi : V = W|_{X_2} \to \widetilde{V} = W'|_{X_2}$ be a morphism in $\mathscr{P}(X, D)$. Let $U = U^-(X_2)$. To prove that $\mu_{W,W'}$ is exhaustive we use the following strategy: first we define a $\mathbb{C}^\times$-equivariant map $\psi : W|_U \to W'|_U$ which extends ϕ, and then we prove that ψ extends to a $\mathbb{C}^\times$-equivariant map $\psi : W \to W'$. It is in the second step that we use that ϕ is a morphism in $\mathscr{P}(X, D)$.

We define ψ as follows: for any $z \in U$,

$$\psi_z := (\rho^z_{W'})^{-1} \circ \phi_{z_-} \circ \rho^z_W : W_z \to W'_z. \tag{4.3}$$

Let $X_D^1 = X_D \setminus X_1$. By Corollary 2.6, in order to prove that ψ extends to the whole X_D it suffices to check that ψ extends to X_D^1.

Recall that $p : X_D \to X$ denotes the projection. We are going to prove that for any $z \in D \subset X$ there exists a neighbourhood $z \in B \subset X$ such that $\psi|_{p^{-1}(B)\cap U}$ extends to a map from $W|_{p^{-1}(B)\cap X_D^1}$ to $W'|_{p^{-1}(B)\cap X_D^1}$ This will be done by taking metrics on $W|_{p^{-1}(B)}$ and $W'|_{p^{-1}(B)}$, and by checking that the L^∞ norm of $\psi|_{p^{-1}B\cap U}$ is bounded. The fact that ψ extends will then follow from Riemann's extension theorem. Note that since $p^{-1}(B) \cap U$ is dense in $p^{-1}(B)$ such extension must be unique, and this implies automatically that the extensions of $\psi|_{p^{-1}B\cap U}$ for different choices of U patch together, and that the resulting extension of ψ is $\mathbb{C}^\times$-equivariant.

4.3.1. Take a point $z \in D$ and coordinates $z_1, \dots, z_n$ in a neighbourhood $B' \subset X$ of z such that

1. z corresponds to $(0, \dots, 0)$,
2. D is given by $z_1 = 0$, and
3. $c = (z_1, \dots, z_n) : B' \to \mathbb{C}^n$ identifies biholomorphically $B(0, 1) \subset \mathbb{C}^n$ with a neigbourhood $B \subset B'$ of z (here $B(a, r)$ is the ball of radius r centered at $a \in \mathbb{C}^n$).

Then there is a $\mathbb{C}^\times$-equivariant isomorphism $\nu : p^{-1}(B) \simeq p_n^{-1}B(0, 1) \subset H_n$ and we define $U' := U_n \cap p_n^{-1}B(0, 1)$ (see Subsection 3.1 for the definitions of $p_n : H_n \to \mathbb{C}^n$ and $U_n \subset H_n$). Let us transport the vector bundles W, W' to $p^{-1}B(0, 1)$ by defining

$$Y = (\nu^{-1})^* W|_{p^{-1}B}, \qquad Y' = (\nu^{-1})^* W'|_{p^{-1}B}.$$

By a slight abuse of notation, $\psi : Y|_{U'} \to Y'|_{U'}$ will be the corresponding map. Take metrics on Y and Y' which are invariant under the action of $S^1 \subset \mathbb{C}^\times$.

4.3.2. Let $\frac{1}{2} > \epsilon > \delta^2 > 0$ be small enough so that we have $\mathbb{C}^\times$-equivariant trivialisations as the ones given by Lemma 2.1, and centered around $z = (0, \dots, 0, [0 : 0 : 1])$, of the restrictions of Y, Y' to the open subset $C = \xi^{-1}([-1 - \delta, -1 + \delta]) \cap p_n^{-1}B(0, 2\epsilon)$ of H_n. Let $S \subset C$ be the neighbourhood of z defined as follows

$$S := \{(t_1, \dots, t_n, [x : y : 1]) \mid \sum_{i=1}^n |t_i|^2 \le \epsilon^2,\ 0 \le |x|, |y| \le \delta,\ t_1 = xy\} \subset p_n^{-1}B(0, 1).$$

Lemma 4.5. *The restriction of ψ to $S \cap U'$ has bounded L^∞ norm.*

Proof. In all the proof t will denote the n-uple $(t_1, \dots, t_n)$. Note that we have $S \cap U' = \{(t, [x : y : 1]) \in S \mid x \neq 0\}$. Let $Y_z = Y(0) \oplus Y(1)$ and $Y'_z = Y'(0) \oplus Y'(1)$ be the decompositions in weights of the fibres of Y and Y' over z. Then, since $S \subset C$, we have weakly $\mathbb{C}^\times$-equivariant trivialisations

$$Y|_S \simeq S \times (Y(0) \oplus Y(1)) \qquad \text{and} \qquad Y'|_S \simeq S \times (Y'(0) \oplus Y'(1)).$$

Using these trivialisations, the restriction of ψ to $S \cap U'$ can be written in matrix form as

$$\psi = \begin{pmatrix} \psi_{11} & \psi_{01} \\ \psi_{10} & \psi_{00} \end{pmatrix},$$

where we view ψ_{ij} as a weakly $\mathbb{C}^{\times}$-equivariant map from $S \cap U'$ to $\mathrm{Hom}(Y(i), Y'(j))$. This means that if $\theta \in \mathbb{C}^{\times}$, $(t, [x : y : 1]) \in S$ and $(t, [\theta x : \theta^{-1} y : 1]) \in S$, then

$$\psi_{ij}(t, [\theta x : \theta^{-1} y : 1]) = \theta^{i-j} \psi_{ij}(t, [x : y : 1]). \tag{4.4}$$

Let $S_\delta = \{(t, [x : y : 1]) \in S \mid x = \delta\}$.

Let $S_{\delta,0} = \{(t, [x : y : 1]) \in S_\delta \mid y = 0\}$, and observe that $S_{\delta,0} \subset U^+(z)$. Since the map ϕ which was used to construct ψ belonged to $\mathscr{P}(X, D)$, we deduce that $\psi(W^{+,1}(X_0)) \subset W'^{+,1}(X_0)$. The restrictions to $S_{\delta,0}$ of the pull backs $\nu^* W^{+,1}(X_0)$ and $\nu^* W'^{+,1}(X_0)$ correspond, using the trivialisations of Y and Y', to $S_{\delta,0} \times Y(1)$ and $S_{\delta,0} \times Y'(1)$ respectively, and from this it follows that

$$\psi_{10}|_{S_{\delta,0}} = 0. \tag{4.5}$$

It now follows from (4.5) and the facts that ψ is smooth and $S_\delta \subset S \cap U'$ is compact, that there is a constant K such that for any $(t, [\delta : y : 1]) \in S_\delta$ we have

$$|\psi_{10}(t, [\delta : y : 1])| \leq K|y| \tag{4.6}$$

$$|\psi_{ij}(t, [\delta : y : 1])| \leq K, \tag{4.7}$$

for any $(i, j) \subset \{0, 1\}^2$. Now, combining (4.4) with (4.6) and (4.7) we deduce that $\psi|_{S \cap U'}$ has bounded L^∞ norm. Indeed, let $(t, [x : y : 1]) \in S \cap U'$. Putting $\theta = \delta x^{-1}$ in (4.4) and taking norm we get, using (4.6):

$$|\psi_{10}(t, [x : y : 1])| = |\delta x^{-1} \psi_{10}(t, [\delta : yx\delta^{-1} : 1])| \leq |\delta x^{-1} K y x \delta^{-1}| = |Ky| \leq K\delta.$$

(This makes sense because $(t, [\delta : yx\delta^{-1} : 1]) \in S_\delta$, since $|yx\delta^{-1}| \leq \delta$.) Finally, if $(1, 0) \neq (i, j) \in \{0, 1\}^2$ and $(t, [x : y : 1]) \in S \cap U'$ we obtain, using (4.4) with $\theta = \delta x^{-1}$ and (4.7):

$$|\psi_{ij}(t, [x : y : 1])| = |(\delta x^{-1})^{i-j} \psi_{ij}(t, [\delta : yx\delta^{-1} : 1])| \leq |\psi_{ij}(t, [\delta : yx\delta^{-1} : 1])| \leq K,$$

since then $i - j \leq 0$, so $|x| \leq \delta$ implies $|\delta x^{-1}|^{i-j} \leq 1$. □

4.3.3. Let $0 < \delta' < \delta$ and $0 < \epsilon' < \epsilon$ be small enough so that $C' = \xi^{-1}([-1 - \delta', -1 + \delta']) \cap p_n^{-1} B(0, \epsilon') \subset S$.

Lemma 4.6. *For any $0 > \alpha > -1 + \delta'$ the restriction of ψ to $\xi^{-1}([-2, \alpha]) \cap p_n^{-1} B(0, \epsilon') \cap U$ has bounded L^∞ norm.*

Proof. Let $C_1 := \xi^{-1}([-2, -1 - \delta']) \cap p_n^{-1} B(0, \epsilon')$. Since $\overline{C_1} \subset U$, it follows from compactness that $|(\psi|_{C_1})|_{L^\infty} < \infty$. By assumption, $C_2 := C' \cap U \subset S$, so Lemma 4.5 implies that $|(\psi|_{C_2})|_{L^\infty} < \infty$. It remains to prove that $|(\psi|_{C_3})|_{L^\infty} < \infty$, where $C_3 = \xi^{-1}([-1 + \delta', \alpha]) \cap p_n^{-1} B(0, \epsilon') \cap U$. Given $z \in \overline{C_3}$, let $\lambda(z) \in \mathbb{R}$ be defined by the condition $\xi(\lambda(z) \cdot z) = -1 + \delta$ (this is well defined because the elements of $\overline{C_3}$ have trivial stabiliser). By compactness of $\overline{C_3}$ it follows that $\lambda : C_3 \to \mathbb{R}$ is bounded. Finally,

since $\xi^{-1}(-1+\delta)\cap C_3 \subset C_2$ and we already know that $|(\psi|_{C_2})|_{L^\infty} < \infty$, it follows from the $\mathbb{C}^\times$-equivariance of ψ and the boundedness of λ that $|(\psi|_{C_3})|_{L^\infty} < \infty$. The lemma follows now because

$$\xi^{-1}([-2,\alpha]) \cap p_n^{-1}B(0,\epsilon') \cap U = C_1 \cup C_2 \cup C_3.$$

$\square$

This lemma implies, by making $\alpha \to 0$, that the section ψ extends to X_D^1, and this finishes the proof of the exhaustivity of $\mu_{W,W'}$.

4.4. **Constructing objects of $\mathscr{V}_{\mathbb{C}^\times}(X,D)$ from objects of $\mathscr{P}(X,D)$.** Let $\mathscr{L} \to X_D$ be the line bundle associated to the divisor Δ^-, with the lift of the action of $\mathbb{C}^\times$ whose existence is granted by Lemma 2.9. By construction it follows that there is a $\mathbb{C}^\times$-equivariant isomorphism $\mathscr{L}|_{\Delta^-} \simeq N^-$, so from Lemma 3.3 we deduce that $\mathscr{L}$ is an object of $\mathscr{V}_{\mathbb{C}^\times}(X_D)$ (that the weight of the restriction $\mathscr{L}|_{X_2}$ is zero follows from $\Delta^- \cap X_2 = \emptyset$, see Lemma 2.9). Finally, it follows from the construction that there is a nowhere zero section $\psi \in H^0(X_2;\mathscr{L}|_{X_2})$ (again, because $X_2 \cap \Delta^- = \emptyset$).

Let us take an object $(V,V_1) \in \mathscr{P}(X,D)$. Our aim is to find an equivariant bundle $W \in \mathscr{V}_{\mathbb{C}^\times}(X_D)$ such that $\mu(W) \simeq (V,V_1)$. Let $E = E_0 \oplus E_1$ be a direct sum of vector spaces satisfying $\dim E = \operatorname{rk} V$ and $\dim E_1 = \operatorname{rk} V_1$. Let $\{A_q\}_{q\in I}$ be a family of open subsets of X, which cover X, and such that for any $q \in I$ there is a vector bundle isomorphism $\phi_q : V|_{A_q} \to A_q \times E$ such that $\phi_q(V_1|_{A_q\cap D}) = (A_q \cap D)\times E_1$. Let us define for any q, q' the following sets:

$$D_q = A_q \cap D, \qquad A_{q,q'} = A_q \cap A_{q'}, \qquad D_{q,q'} = A_q \cap A_{q'} \cap D.$$

The isomorphism class of (V,V_1) is completely determined by the transition functions

$$\{\phi_{q,q'} = \phi_{q'}|_{A_q\cap A_{q'}} \circ (\phi_q|_{A_q\cap A_{q'}})^{-1} \mid q,q' \in I\}.$$

Note that the covering $\{A_q\}$ of X induces a covering $\{(A_q)_{D_q}\}$ of X_D. To save on typing, we will denote by p the restriction of the projection map $p : X_D \to X$ on any $(A_q)_{D_q}$. We define for any $q \in I$ the following $\mathbb{C}^\times$-equivariant bundle on $(A_q)_{D_q}$:

$$M_q := \underline{E_0} \oplus \underline{E_1} \otimes \mathscr{L}|_{(A_q)_{D_q}}$$

(here $\underline{E_0}$ and $\underline{E_1}$ denote the trivial bundles on $(A_q)_{D_q}$ with fibres E_0, E_1 respectively, and with the trivial lift of the action of $\mathbb{C}^\times$). Then M_q is an object of $\mathscr{V}_{\mathbb{C}^\times}((A_q)_{D_q})$. To get an equivariant bundle on X_D by patching the bundles M_q we have to specify transition functions

$$\{\Phi_{q,q'} : M_q|_{(A_q)_{D_q}\cap(A_{q'})_{D_{q'}}} \to M_{q'}|_{(A_q)_{D_q}\cap(A_{q'})_{D_{q'}}} \mid q,q' \in I\}.$$

The section ψ defined above provides, for any q, and isomorphism

$$\eta'_q : A_q \times E \to M_q|_{X_2\cap(A_q)_{D_q}}$$

(here we use the identification $X_2 \simeq X$ to identify $A_q \simeq X_2 \cap (A_q)_{D_q}$), which induces an isomorphism of objects in $\mathscr{P}(A_q, D_q)$

$$\eta_q : (A_q \times E, D_q \times E_1) \to \mu(M_q).$$

Now, to define $\Phi_{q,q'}$ we will use the fact that the map

$$\mu_{q,q'} : \mathrm{Mor}_{\mathscr{V}_{\mathbb{C}^\times}((A_q)_{D_q}\cap(A_{q'})_{D_{q'}})}(M_q, M_{q'}) \to \mathrm{Mor}_{\mathscr{P}(A_{q,q'},D_{q,q'})}(\mu(M_q, M_{q'})$$

induced by the functor μ is an isomorphism (this has been proved in Sections 4.2 and 4.3). Namely, we set:

$$\Phi_{q,q'} := \mu^{-1}_{q,q'}(\eta_{q'} \circ \phi_{q,q'} \circ \eta_q^{-1}).$$

Then the functions $\{\Phi_{q,q'}\}$ are all $\mathbb{C}^\times$-equivariant and they satisfy the cocycle condition (because the functions $\{\phi_{q,q'}\}$ satisfy it), so they define an equivariant bundle $W \to X_D$. The isomorphisms $\{\eta_q\}$ and $\{\phi_q\}$ give an isomorphism between $\mu(W)$ and (V, V_1), and we are done.

4.5. **An equivariant version of Theorem 4.2.** Suppose that a group G acts on X preserving the divisor D. By Lemma 3.1 such an action lifts to X_D and the resulting action commutes with the action of $\mathbb{C}^\times$. Furthermore, $X_0 \cup X_1 \cup X_2 \subset X_D^G$. Consider the following categories (which are the G-equivariant versions of the categories $\mathscr{V}_{\mathbb{C}^\times}(X_D)$ and $\mathscr{P}(X, D)$):

1. $\mathscr{V}^G_{\mathbb{C}^\times}(X_D)$ is the category whose objects are $G \times \mathbb{C}^\times$-equivariant vector bundles $W \to X_D$ which, considered as a $\mathbb{C}^\times$-equivariant bundle, belong to $\mathscr{V}_{\mathbb{C}^\times}(X_D)$, and whose morphisms are the $G \times \mathbb{C}^\times$-equivariant morphisms of vector bundles.
2. $\mathscr{P}^G(X, D)$ is the category whose objects are pairs (V, V_1), where $V \to X$ is a G-equivariant vector bundle and $V_1 \to D$ is a G-invariant subbundle of $V|_D$, and whose morphisms are the G-equivariant morphisms of bundles which preserve the subbundles.

Note that we have obvious functors $f_{\mathcal{V}} : \mathscr{V}^G_{\mathbb{C}^\times}(X_D) \to \mathscr{V}_{\mathbb{C}^\times}(X_D)$ and $f_{\mathscr{P}} : \mathscr{P}^G(X, D) \to \mathscr{P}(X, D)$ which forget the action of G. We now construct a functor $\mu^G : \mathscr{V}^G_{\mathbb{C}^\times}(X_D) \to \mathscr{P}^G(X, D)$ as follows. The action of μ^G on objects maps and equivariant bundle $W \to \mathscr{V}^G_{\mathbb{C}^\times}(X_D)$ to the pair $\mu^G(W) = (V, V_1) \in \mathscr{P}^G(X, D)$ satisfying: (1) $f_{\mathscr{P}}(V, V_1) = \mu(f_{\mathcal{V}} W)$ and (2) the lift of G to V is the one obtained by identifying $V = W|_{X_2}$ (recall that we have canonically $X_2 \simeq X$). Finally, the action of μ^G on morphisms is by taking the restriction to X_2 (exactly like that of μ).

Theorem 4.7. *The functor μ^G induces an equivalence of categories.*

Proof. This follows essentially from Theorem 4.2 and the commutativity of diagram (4.2) applied to the biholomorphisms $f \in \mathrm{Bihol}(X)$ given by the action of the elements of G on X. The only thing which might not be clear is the fact that any object in $\mathscr{P}^G(X, D)$ is isomorphic to the image by μ^G of some object in $\mathscr{V}^G_{\mathbb{C}^\times}(X_D)$. Let us clarify this point. Giving a lift of the G action on X to a vector bundle $V \to X$ is the same thing as giving a set of isomorphisms

$$I = \{i_g : V \to l_g^* V \mid g \in G\}$$

(where $l_g : X \to X$ denotes the biholomorphism induced by the action of $g \in G$) which satisfy certain cocycle condition (let us call it G-condition). If $(V, V_1) \in \mathscr{P}^G(X, D)$, the isomorphisms in I are also compatible with V_1. Hence, if we have $W \in \mathscr{V}_{\mathbb{C}^\times}(X_D)$ and an

isomorphism $\mu(W) \simeq V$, we can use Theorem 4.2 to lift the family of isomorphisms I to a family of $\mathbb{C}^\times$-equivariant isomorphisms

$$I' = \{i_g : W \to l_g'^* W \mid g \in G\},$$

denoting $l_g' : X_D \to X_D$ the action of $g \in G$. Finally, again by Theorem 4.2, the elements in I' satisfy the G-condition, precisely because the elements in I satisfy it. ☐

In the following lemmae, recall that $N^- \to \Delta^-$ and $N^+ \to \Delta^+$ are the normal bundles of the inclusions $\Delta^- \subset X_D$ and $\Delta^+ \subset X_D$. By an abuse of notation, we will denote by $L \to D$ be the normal bundle of the inclusion $D \subset X$.

Lemma 4.8. *Suppose that G is reductive. Let $a \in X^G$, $W \in \mathscr{V}^G_{\mathbb{C}^\times}(X_D)$ and $(V, V_1) := \mu^G(W)$. Let $b \in p^{-1}(a) \cap X_D^G$. If $b \notin \Delta^-$ (resp. if $b \in \Delta^-$) then there is a G-equivariant isomorphism of fibres $W_b \simeq V_a$ (resp. $W_b \simeq (V_1)_a \otimes N_b^- \oplus (V_a/(V_1)_a)$).*

Proof. Since by assumption G is reductive and everything is holomorphic, we can use Weyl's unitary trick and work with a maximal compact subgroup of G, which by an abuse of notation we will denote, only in this proof, by the same symbol G. If $a \notin D$ then the result follows easily from Lemma 2.4. Now assume that $a \in D$. Using the same idea as in the proof of Lemma 2.1 and up to shrinking X to a small G-invariant neighbourhood of a we may assume that there is a G-equivariant trivialisation

$$V \simeq (X \times (V_1)_a) \oplus (X \times V_a/(V_1)_a)$$

inducing a trivialisation $V_1 \simeq D \times (V_1)_a$. Let $\mathscr{L} \to X_D$ be the line bundle corresponding to the divisor Δ^- together with the lift of the $G \times \mathbb{C}^\times$ action induced by Lemma 2.9. Then

$$W_0 := (X_D \times V_a') \otimes \mathscr{L} \oplus (X_D \times (V_a/(V_1)_a))$$

with the corresponding lifts of the $G \times \mathbb{C}^\times$ action satisfies $\mu^G(W_0) = (V, V_1)$, so by Theorem 4.7 we have a $G \times \mathbb{C}^\times$-equivariant isomorphism $W_0 \simeq W$. Now the result follows by using the second part of Lemma 2.9 to identify $\mathscr{L}_b \simeq N_b^-$ as representations of G. ☐

Lemma 4.9. *For any $b \in X_D^G \cap \Delta^-$ there is a G-equivariant isomorphism $N_b^- \simeq L_{p(b)}$.*

Proof. Let $a = p(b)$ and identify $\mathbb{P}(\mathbb{C}^3) \simeq \mathbb{P}(L \oplus \underline{\mathbb{C}} \oplus \underline{\mathbb{C}})_a$. Since $p^{-1}(a) \cap \Delta^- \cap X_D^G$ is either $\{[0:1:0]\} \cup \{[0:0:1]\}$ or $\{[0:x:y] \mid (x,y) \in \mathbb{C}^2 \setminus (0,0)\}$ (depending on whether the representation L_a of G is trivial or not), it suffices to prove the result for $b = [0:1:0] \in X_1$ or $b = [0:0:1] \in X_2$. For the first case, observe that the restriction of N^- to $X_1 \cap \Delta^-$ is isomorphic to L (because X_1 and Δ^- intersect transversely along a submanifold which can be canonically identified — using p — with D). If $b = [0:0:1]$, then observe that the map $f : L_a \to X_D$ which sends $x \in L_a$ to $[x:0:1]$ is equivariant and transverse to Δ^- at b. Hence it gives an equivariant identification of L_a with N_b^-. On the other hand, by Lemma 2.9 the representation L_a of G is isomorphic to L_a, and we are done. ☐

Lemma 4.10. *Let $b \in X_D^G \cap \Delta^+$. If $b \in X_0$ then the representation N_b^+ of G is the trivial one, and if $b \in X_2$ then the representation N_b^+ is isomorphic to $L_{b(p)}$.*

Proof. Exactly the same as that of the previous lemma. □

5. Parabolic structures over a smooth divisor

In this section X will be a manifold and $D \subset X$ a smooth divisor. Let us fix an integer $r \geq 1$. Let $\mathscr{P}(X, D, r)$ be the category defined as follows:

1. The objects of $\mathscr{P}(X, D, r)$ are pairs $(V, \mathcal{V})$ consisting of a vector bundle $V \to X$ and a filtration $\mathcal{V}$ of vector bundles over D:
$$\mathcal{V} = (V_1 \subset \cdots \subset V_r \subset V|_D)$$
(note that the inclusions need not be strict).
2. The morphisms in $\mathscr{P}(X, D, r)$ between two objects $(V, \mathcal{V})$ and $(V', \mathcal{V}')$ are the morphisms of vector bundles $\phi : V \to V'$ whose restriction to D respects the filtrations $\mathcal{V}$ and $\mathcal{V}'$, i.e., for any $1 \leq j \leq r$, $\phi|_D(V_j) \subset V_j'$.

Our aim in this section is to obtain, by applying recursively the construction of Section 3, a $(\mathbb{C}^\times)^r$-manifold $X(D, r)$ which fibers over X, and to prove that the category $\mathscr{P}(X, D, r)$ is equivalent to a full subcategory of the category of $(\mathbb{C}^\times)^r$-equivariant bundles over $X(D, r)$.

5.1. **Notations and definitions.**

5.1.1. *Weights of* $(\mathbb{C}^\times)^r$. The objects of the subcategory of equivariant vector bundles which will ultimatelly be equivalent to $\mathscr{P}(X, D, r)$ will be those which satisfy a certain restriction on the weights of a action of $(\mathbb{C}^\times)^r$) (to be defined below). In order to be able to specify this restriction we need to introduce some notation on weights.

We identify the characters of $(\mathbb{C}^\times)^r$ with $\mathbb{Z}^r$ by assigning to $a = (a_1, \dots, a_r) \in \mathbb{Z}^r$ the character
$$\chi_a : (\mathbb{C}^\times)^r \ni (\theta_1, \dots, \theta_r) \mapsto \theta_1^{a_1} \cdot \cdots \cdot \theta_r^{a_r} \in \mathbb{C}^\times.$$
If $r < s$ we map $\mathbb{Z}^r \to \mathbb{Z}^s$ by sending $(a_1, \dots, a_r)$ to $(a_1, \dots, a_r, 0, \dots, 0)$, and in this way we will sometimes implicitely view the set of characters of $(\mathbb{C}^\times)^r$ as subset of the characters of $(\mathbb{C}^\times)^s$. In particular, $0 \in \mathbb{Z}^r$ will denote $(0, \dots, 0)$.

For any $1 \leq j \leq r$ let $\pi_j : \mathbb{Z}^r \to \mathbb{Z}$ be the projection to the j-th factor. Let e_j be the character corresponding to the projection $(\mathbb{C}^\times)^r \to \mathbb{C}^\times$ to the j-th factor. Note that $e_1, \dots, e_r$ is the canonical basis of $\mathbb{Z}^r$ and that $\pi_i(e_j) = \delta_{ij}$. We also define for any $1 \leq j \leq r$
$$f_j := e_j + 2e_{j-1} + \sum_{i \geq 2} e_{j-i},$$
(where we understand that $e_k = 0$ whenever $k \leq 0$).

Let $\Pi_{j+1} : \mathbb{Z}^{j+1} \to \mathbb{Z}^j$ be defined as

$$\Pi_{j+1}(f) := \begin{cases} f & \text{if } \pi_{j+1}(f) = 0 \\ f - \pi_{j+1}(f) - e_{j+1} - e_j + e_{j+1} & \text{if } \pi_{j+1}(f) \neq 0. \end{cases}$$

The following equalities follow immediately from the definition:

$$\Pi_{j+1}(f_{j+1}) = f_j \qquad \text{and} \qquad \Pi_{j+1}(f_i) = f_i \text{ for any } i \leq j. \tag{5.8}$$

5.1.2. *The iterated construction.* We define recursively a sequence of manifolds

$$Y_0, Y_1, \ldots, Y_r$$

with projections $p_j : Y_j \to Y_{j-1}$ and a smooth divisor $\Delta_j^+ \subset Y_j$ for any $1 \leq j \leq r$ as follows. We first set $Y_0 := X$ and $\Delta_0^+ := D$. If $0 \leq j < r$, we apply the construction of Subsection 3 to define

$$p_{j+1} : Y_{j+1} := (Y_j)_{\Delta_j^+} \to Y_j.$$

By construction the manifold Y_{j+1} carries an action of $G_j := \mathbb{C}^\times$, whose fixed point locus is the disjoint union of two copies of Y_j, denoted by $Y_{j,1}, Y_{j,2}$, and a copy of Δ_j^+, denoted by $Y_{j,0}$. With this in mind, we define

$$\Delta_{j+1}^+ := \overline{U^+(Y_{j,0})} \qquad \text{and} \qquad \Delta_{j+1}^- := \overline{U^-(Y_{j,0})}.$$

Both Δ_{j+1}^+ and Δ_{j+1}^- are smooth divisors in of Y_{j+1}.

Let $i \in \{0, 1, 2\}$. We define $Y_1[i] := Y_{0,i}$ and, for any $1 \leq j < r$, we set $Y_{j+1}[i] := Y_{j,i} \cap p_{j+1}^{-1}(Y_j[i]) \subset Y_{j+1}$.

If $1 \leq j < r$, we may use Lemma 3.1 to lift the action of G_j on Y_j to an action on Y_{j+1}, which commutes with the action of G_{j+1}. And, using recursion, we get, for any $1 \leq j \leq r$, a natural action of $G(j) = G_1 \times \cdots \times G_j$ on Y_j. It is easy to check that for any $i \in \{0, 1, 2\}$ the submanifold $Y_j[i]$ belongs to the fixed point set of the action of $G(j)$.

Definition 5.1. *We define $X(D, r)$ to be Y_r and, for any $i \in \{0, 1, 2\}$, $X(D, r)[i] := Y_r[i]$.*

By construction we have a tower of manifolds

$$X(D, r) = Y_r \xrightarrow{p_r} Y_{r-1} \xrightarrow{p_{r-1}} \cdots \xrightarrow{p_1} Y_0 = X \tag{5.9}$$

and, identifying for any $0 \leq j < r$ the submanifold Y_j with $Y_{j,2} \subset Y_{j+1}$ we get a chain of inclusions

$$X = Y_0 \subset Y_1 \subset \cdots \subset Y_{r-1} \subset Y_r = X(D, r). \tag{5.10}$$

These inclusions induce a chain of equalities

$$X = Y_1[2] = Y_2[2] = \cdots = Y_r[2]. \tag{5.11}$$

Lemma 5.2. *For any $i \in \{0, 1, 2\}$ the following is a strong inclusion (see Subsection 2.1): $Y_{j+1}[i] \subset Y_{j,i}$.*

Proof. Easy from the definitions. ∎

5.1.3. *Categories of equivariant bundles.* Let $\mathscr{V}_{G(r)}(X(D,r))$ be the category defined as follows.

1. The objects of $\mathscr{V}_{G(r)}(X(D,r))$ are $G(r)$-equivariant bundles $W \to X(D,r)$ whose only $G(r)$-weight on $X(D,r)[1]$ and $X(D,r)[2]$ is zero and whose $G(r)$-weights on $X(D,r)[0]$ belong to $\{f_1,\dots,f_r\}$.
2. The morphisms in $\mathscr{V}_{G(r)}(X(D,r))$ between to objects W and W' are the $G(r)$-equivariant maps of vector bundles $\psi : W \to W'$.

Our aim is to construct a functor $\mu(r) : \mathscr{V}_{G(r)}(X(D,r)) \to \mathscr{P}(X,D,r)$ inducing an equivalence of categories. For that we will construct a set of auxiliar categories $\mathscr{VP}_0,\dots,\mathscr{VP}_r$ making up a bridge from $\mathscr{V}_{G(r)}(X(D,r))$ to $\mathscr{P}(X,D,r)$.

Let $0 \le j \le r$. Define the category $\mathscr{VP}_j$ as follows.

1. The objects of $\mathscr{VP}_j$ are pairs $(W^j,\mathcal{W}^j)$ consisting of a $G(j)$-equivariant bundle $W^j \to Y_j$ and a filtration of bundles over Δ_j^+:
$$\mathcal{W}^j = (W_1^j \subset \cdots \subset W_{r-j}^j \subset W^j|_{\Delta_j^+}),$$
subject to the following constraints: the only $G(j)$-weight of W^j on $Y_j[1]$ and $Y_j[2]$ is zero, the $G(j)$-weights of W^j on $Y_j[0]$ belong to $\{f_1,\dots,f_j\}$ and, for any $1 \le s \le r-j$, the only $G(j)$-weight of W_s^j on $Y_j[0]$ is f_j (in fact this condition holds for any $1 \le s \le r-j$ if and only if it holds for $s = r-j$).
2. The morphisms in $\mathscr{VP}_j$ between two objects $(W^j,\mathcal{W}^j)$ and $(W'^j,\mathcal{W}'^j)$ are the $G(j)$-equivariant maps of vector bundles $\phi : W^j \to W'^j$ whose restriction to Δ_j^+ respects the filtrations $\mathcal{W}^j$ and W'^j.

Observe that $\mathscr{VP}_r = \mathcal{V}_{G(r)}(X(D,r))$ and $\mathscr{VP}_0 = \mathscr{P}(X,D,r)$. In the next subsection we will construct for any $0 \le j < r$ a functor $\mu_{j+1} : \mathscr{VP}_{j+1} \to \mathscr{VP}_j$, and we will define $\mu(r)$ to be the composition $\mu(r) := \mu_1 \circ \mu_1 \circ \cdots \circ \mu_r$. Finally, we will show that each μ_j induces an equivalence of categories. Hence, $\mu(r)$ gives also an equivalence of categories.

5.2. **The functors μ_{j+1}.** Let us fix some $0 \le j < r$. Our aim here is to define a functor
$$\mu_{j+1} : \mathscr{VP}_{j+1} \to \mathscr{VP}_j.$$

5.2.1. Fist of all, we define the action of μ_{j+1} on morphisms to be the restriction to $Y_j \subset Y_{j+1}$.

We now define μ_{j+1} acting on objects. Let $(W^{j+1},\mathcal{W}^{j+1}) \in \mathscr{VP}_{j+1}$, where $\mathcal{W}^{j+1} = (W_1^{j+1} \subset \cdots \subset W_{r-j-1}^{j+1})$. By definition, W^{j+1} is a vector bundle over $Y_{j+1} = (Y_j)_{\Delta_j^+}$. We can now use the functor $\mu^{G(j)}$ defined in Subsection 4.5 to define
$$(W^j, W_{r-j}^j) := \mu^{G(j)}(W^{j+1}).$$
On the other hand, recall that we have an inclusion $Y_j \subset Y_{j+1}$ by identifying $Y_j = Y_{j,2} \subset Y_{j+1}$. In this way we get an identification between Δ_j^+ and $\Delta_{j+1}^+ \cap Y_{j,2}$. With this in

mind, we define, for any $1 \leq s \leq r-j-1$,

$$W_s^j := W_s^{j+1}|_{\Delta_{j+1}^+ \cap Y_{j,2}},$$

and we set $\mathcal{W}^j := (0 \subset W_1^j \subset \cdots \subset W_{r-j}^j)$. Then, we define

$$\mu_{j+1}(W^{j+1}, \mathcal{W}^{j+1}) := (W^j, \mathcal{W}^j).$$

5.2.2. Let us check that $(W^j, \mathcal{W}^j)$ is indeed an object of $\mathscr{VP}_j$. To begin with, observe that, since both $Y_{j,2}$ and Δ_{j+1}^+ are $G(j)$ invariant submanifolds of Y_{j+1}, all the bundles $W^j, W_1^j, \ldots, W_{r-j}^j$ inherit an action of $G(j)$.

We also need to check that $W_{r-j-1}^j \subset W_{r-j}^j$. By assumption, for any $1 \leq s \leq r-j-1$, the only $G(j+1)$-weight of W_s^j on $Y_{j+1}[0]$ is f_{j+1}. This implies that the only G_{j+1}-weight of W_s^j on $Y_{j+1}[0]$ is 1. Indeed, since $Y_{j+1}[0] \subset Y_{j,0}$ is a strong inclusion (see Lemma 5.2) and $Y_{j,0} \subset Y_{j+1}^{G_{j+1}}$, we know that the only G_{j+1}-weight of W_s^j on $Y_{j,0}$ is 1. Now, Lemma 4.1 implies that $W_{r-j-1}^j \subset W_{r-j}^j$.

In remains to check that the weights of W^j restricted to $Y_j[i]$ for $i = 0, 1, 2$ are the right ones. This is proved in the following lemma.

Lemma 5.3. *Let $(W^j, W_{r-j}^j) = \mu^{G(j)}(W^{j+1})$. Then the weights of $G(j)$ acting on the restriction of W^j to $Y_j[1]$ (resp. $Y_j[2]$, $Y_j[0]$) are 0 (resp. 0, contained in $\{f_1, \ldots, f_j\}$). Furthermore, $\chi_{G(j)}(W_{r-j}^j|_{Y_j[0]}) = f_j$.*

Proof. (a) Observe first that $G(j)$ acts trivially on $p_{j+1}^{-1} Y_j[1] \subset Y_{j+1}$. Indeed, the action of $G(j)$ on $Y_j[1] \subset Y_j$ is trivial. And, since $Y_{j,1} \cap \Delta_j^+ = \emptyset$, the lift of the action of $G(j)$ to the line bundle $L_j \to Y_j$ corresponding to the divisor $\Delta^*(j) \subset Y_j$ is trivial (see Lemma 2.9). On the other hand, $Y_{j+1}[1] \subset p_{j+1}^{-1} Y_j[1]$ is a strong inclusion. Hence, since by hypothesis $\chi_{G(j)}(W^{j+1}|_{Y_{j+1}[1]}) = \{0\}$, it follows that $\chi_{G(j)}(W^{j+1}|_{p_{j+1}^{-1} Y_j[1]}) = \{0\}$. Finally, by definition $W^j|_{Y_j[1]} = W^{j+1}|_{p_{j+1}^{-1} Y_j[1] \cap Y_{j,2}}$ as $G(j)$-equivariant bundles, so that $\chi_{G(j)}(W^j|_{Y_j[1]}) = 0$.

(b) By (5.11) and the definition of W^j we have $\chi_{G(j)}(W^j|_{Y_j[2]}) = \chi_{G(j)}(W^{j+1}|_{Y_{j+1}[2]}) = 0$, where the second equality follows from our hypothesis.

(c) For any $1 \leq s \leq r$, let $N_j^- \to \Delta_j^-$ and $N_j^+ \to \Delta_j^+$ be the normal bundles of the inclusions $\Delta_j^- \subset Y_j$ and $\Delta_j^+ \subset Y_j$ respectively. Note that $N_j^+ = L_j|_{\Delta_j^+}$, where $L_j \to Y_j$ is the line bundle corresponding to $\Delta_j^+ \subset Y_j$. We will now compute the weight of each group G_i acting on $N_j^-|_{Y_j[0]}$.

1. Suppose that $i < j-1$. Then $\chi_{G_i}(N_j^-|_{Y_j[0]}) = \chi_{G_i}(N_{j-1}^+|_{Y_{j-1}[0]}) = 0$, where the first equality follows from by Lemma 4.9, and the second one from $Y_{j-1}[0] \cap Y_{j-2,2} = \emptyset$.
2. The case $i = j-1$. We have $\chi_{G_{j-1}}(N_j^-|_{Y_j[0]}) = \chi_{G_{j-1}}(N_{j-1}^+|_{Y_{j-1}[0]}) = -1$, where the first equality follows from Lemma 4.9, and the second one from $\chi_{G_{j-1}}(N_{j-1}^+|_{Y_{j-2,0}}) = \{-1\}$ (Lemma 3.2) and $Y_{j-1}[0] \subset Y_{j-2,0}$.

3. The case $i = j$. We have $\chi_{G_j}(N_j^-|_{Y_j[0]}) = 1$ since, by Lemma 3.3, $\chi_{G_j}(N_j^-|_{Y_{j-1,0}}) = \{1\}$, and $Y_j[0] \subset Y_{j-1,0}$.

We have thus obtained:

$$\chi_{G(j)}(N_j^-|_{Y_j[0]}) = -e_{j-1} + e_j. \tag{5.12}$$

Let now $a \in Y_j[0]$ and $b := p_{j+1}^{-1}(a) \cap Y_{j,0} \in Y_{j+1}[0]$. Let $\chi_b \subset \mathbb{Z}^{j+1}$ (resp. $\chi'_a \subset \chi_a \subset \mathbb{Z}^j$) be the weights appearing in the decomposition of $(W^{j+1})_b$ (resp. $(W^j_{r-j})_a \subset (W^j)_a$) in irreps of $G(j+1)$ (resp. $G(j)$). Combining Lemma 4.8 with (5.12) we deduce that

$$\chi_a = \Phi_{j+1}(\chi_b) \qquad \text{and} \qquad \chi'_a = \{\Pi_{j+1}(\chi) \mid \chi \in \chi_b,\ \pi_{j+1}(\chi) = 1\}.$$

Now, by assumption $\chi_b \subset \{f_1, \dots, f_{j+1}\}$, so by (5.8) we deduce that $\chi_a \subset \{f_1, \dots, f_j\}$. Finally, if $f \in \{f_1, \dots, f_{j+1}\}$ then $\pi_{j+1}(f) = 1$ if and only if $f = f_{j+1}$. Consequently, again by (5.8), $\chi'_a = \{f_j\}$. This finishes the proof. ☐

Lemma 5.4. *The functor μ_{j+1} induces an equivalence of categories.*

Proof. The proof is completely analogous to that of Theorems 4.2 and 4.7. The key step is to prove that μ_j induces bijections when acting on morphisms, and this can be done following exactly the same steps as in Subsections 4.2 and 4.3. ☐

5.2.3. We sum up in the following theorem all the results which we have obtained so far.

Theorem 5.5. *For any complex manifold X, any smooth divisor $D \subset X$, and integer $r \geq 1$, there exists*

1. *a manifold $X(D, r)$ acted on by $G(r) = (\mathbb{C}^\times)^r$,*
2. *an invariant projection $\pi : X(D, r) \to X$ with a section $\sigma : X \to X(D, r)$,*
3. *a full subcategory $\mathscr{V}_{G(r)}(X(D, r))$ of the category of $G(r)$-equivariant vector bundles on $X(D, r)$, and*
4. *a functor $\mu(r) : \mathscr{V}_{G(r)}(X(D, r)) \to \mathscr{P}(X, D, r)$ inducing an equivalence of categories.*

Proof. The map $\pi : X(D, r) \to X$ is the composition of the maps appearing in (5.9), and the section $\sigma : X \to X(D, r)$ is the composition of the inclusions in (5.10). The fact that $\mu(r)$ induces an equivalence of categories follows from Lemma 5.4. ☐

6. Cohomological questions

In this section we address two different questions. First, that of relating the topology of $W \in \mathscr{V}_\Gamma(X(D, \underline{r}))$ to that of $M(W) = (V, \mathcal{V}_1, \dots, \mathcal{V}_s) \in \mathscr{P}(X, D, \underline{r})$. In particular, we obtain a formula for the first Chern class of W in terms of the first Chern class of V and the ranks of the elements in the filtrations $\mathcal{V}_u$.

Secondly, we compute the Kaehler cone of $X(D, \underline{r})$ in terms of the Kaehler cone of X and the classes $[D_1], \dots, [D_s] \in H_2(X)$. We reduce the computation to the case D smooth and $r = 1$, i.e., to the problem of relating the Kaehler cone of the blow up of $X \times \mathbb{P}^1$ along $D \times \{[0:1]\}$ to the Kaehler cone of X and the class $[D] \in H_2(X)$.

6.1. **Deformations near the divisor.** Let X, X' be manifolds, and let $D \subset X$ and $D' \subset X'$ be smooth divisors. Let $(V, V_1) \in \mathscr{P}(X, D)$ and $(V', V_1') \in \mathscr{P}(X', D')$. We will say that (X, D, V, V_1) and (X', D', V', V_1') are isomorphic near the divisor, and we will write

$$(X, D, V, V_1) \cong (X', D', V', V_1'),$$

if there exist a neighbourhood U (resp. U') of D resp. D'), a biholomorphism $\phi : U \to U'$ which identifies D with D', and an isomorphism $\psi : V|_U \to \phi^* V'|_{U'}$ which identifies V_1 with $\phi^* V_1'$. If $p : X_D \to X$ and $p' : X'_{D'} \to X'$ denote the projections, it follows from Theorem 4.2 that, for any pair of objects $W \in \mathscr{V}_{\mathbb{C}^\times}(X_D)$ and $W' \in \mathscr{V}_{\mathbb{C}^\times}(X'_{D'})$ such that $\mu(W) \simeq (V, V_1)$ and $\mu(W') \simeq (V', V_1')$, ψ induces an isomorphism between $W|_{p^{-1}U}$ and $W'|_{p'^{-1}U'}$ respectively.

We will say that (X, D, V, V_1) and (X', D', V', V_1') are directly deformation equivalent near the divisor, and we will write

$$(X, D, V, V_1) \simeq (X', D', V', V_1'),$$

if there exists a submersion $q : Y \to B$, where B is smooth and connected, a divisor $\Delta \subset Y$ such that $q|_\Delta : \Delta \to B$ is also a submersion, a pair $(Z, Z_1) \in \mathscr{P}(Y, \Delta)$, and two points $b, b' \in B$ such that

$$(q^{-1}b, (q|_\Delta)^{-1}b, Z|_{q^{-1}b}, Z_1|_{(q|_\Delta)^{-1}b}) \cong (X, D, V, V_1)$$

and

$$(q^{-1}b', (q|_\Delta)^{-1}b', Z|_{q^{-1}b'}, Z_1|_{(q|_\Delta)^{-1}b'}) \cong (X', D', V', V_1').$$

We will call deformation equivalence near the divisor, and denote it by $\sim$, the equivalence relation induced by $\simeq$.

Lemma 6.1. *If $(X, D, V, V_1) \sim (X', D', V', V_1')$ and we have $(V, V_1) = \mu(W)$, $(V', V_1') = \mu(W')$ for some $W \in \mathscr{V}_{\mathbb{C}^\times}(X_D)$ and $W' \in \mathscr{V}_{\mathbb{C}^\times}(X'_{D'})$, then there is a C^∞ isomorphism between the restrictions of W and W' to small neighbourhoods of $p^{-1}D$ and $p'^{-1}D'$ respectively.*

Proof. Combine Theorem 4.2 with Ehresmann's Theorem. □

Let $\pi : L \to X$ be the line bundle obtained from D and let $\sigma_0 : X \to L$ be the zero section. Let $\sigma \in H^0(L)$ be a section transverse to σ_0 and such that $\sigma^{-1}(0) = D$. Let $\pi_D : L|_D \to X$ be the restriction of π. The following result will be useful to relate the topology of an object $W \in \mathscr{V}_{\mathbb{C}^\times}(X_D)$ and its image (V, V_1) by μ.

Lemma 6.2. *Let $(V, V_1) \in \mathscr{P}(X, D)$ be any object, and let $V_0 = V|_D/V_1$. Then*

$$(X, D, V, V_1) \sim (L|_D, \sigma_0(D), \pi_D^*(V_0 \oplus V_1), \pi_D^* V_1|_{\sigma_0(D)}).$$

Proof. The proof is split in two steps. We first prove that

$$(X, D, V, V_1) \simeq (L|_D, \sigma_0(D), \pi_D^* V, \pi_D^* V_1|_{\sigma_0(D)}).$$

We use for that the trick of deformation to the normal cone. Let $B = B(0, 2) \subset \mathbb{C}$, and let $B^\times = B \cap \mathbb{C}^\times$. Let $Y^\times$ be the graph of the map $\Sigma : X \times B^\times \to L$ which sends (x, t)

to $t^{-1}\sigma(x)$. Let Y be the closure of $Y^\times$ inside $X\times B\times L$, and let $q:=\pi_2 : Y\to B$ be the projection to the second factor. Let $\Delta\subset Y$ be the divisor $\{(y,t,\sigma_0(y))\mid y\in D,\ t\in B\}$. Let us prove that both q and its restriction to Δ are submersions. This is clearly true in $q^{-1}B^\times = Y^\times$. As for the points in $q^{-1}(0)$, observe that $\pi_1 q^{-1}(0)\subset D$, where $\pi_1 : X\times B\times L\to X$ is the projection. It thus suffices to take for any $x\in D$ any neighbourhood $x\in E\subset X$ and study $\pi_1^{-1}(E)\cap q^{-1}(0)$. So suppose that E is small enough so that there are coordinates $x_1,\dots,x_n$ on E such that x corresponds to $(0,\dots,0)$ and $D=\{x_1=0\}$. Then we can trivialize $L|_E\simeq \underline{\mathbb{C}}$ in such a way that the section σ maps $(x_1,\dots,x_n)$ to x_1. Identifying X with E it follows that $Y^\times=\{(x_1,\dots,x_n,t,t^{-1}x_1)\mid t\in B^\times\}$ (in the last term we just write the fibrewise component of the points in $L|_E$, using the trivialisation). So $Y=\{(x_1,\dots,x_n,t,y)\mid t\in B, ty=x_1\}$. But it is then clear that the map $Y\ni(x_1,\dots,x_n,t,y)\mapsto t\in B$ is a submersion.

Returning to the case of general X, define $Z:=\pi_1^*V\to Y$, and $Z_1:=(\pi_1|_\Delta)^*V'$. Then it is easy to check that

$$(X,D,V,V_1)\cong(q^{-1}(1),q^{-1}(1)\cap\Delta,Z|_{q^{-1}(1)},Z_1|_{q^{-1}(1)\cap\Delta})$$

and that

$$(L|_D,\sigma_0(D),\pi_D^*V,\pi_D^*V_1|_{\sigma_0(D)})\cong(q^{-1}(0),q^{-1}(0)\cap\Delta,Z|_{q^{-1}(0)},Z_1|_{q^{-1}(0)\cap\Delta}).$$

In the second step we prove that

$$(L|_D,\sigma_0(D),\pi_D^*V,\pi_D^*V_1|_{\sigma_0(D)})\simeq(L|_D,\sigma_0(D),\pi_D^*(V_0\oplus V_1),\pi_D^*V'|_{\sigma_0(D)}).$$

This will be a consequence of the following general fact: if $V_1\subset V$ are vector bundles on X and we define $V_0=V/V_1$, then there is a bundle $Z\to Y=X\times\mathbb{C}$ such that $Z|_{X\times\{1\}}\simeq V$ and $Z|_{X\times\{0\}}\simeq V_0\oplus V_1$. Indeed, let us take a C^∞ splitting $V\simeq V_0\oplus V_1$, so that the $\overline{\partial}$-operator of V may be written as

$$\overline{\partial}_V=\begin{pmatrix}\overline{\partial}_{V_0} & \beta\\ 0 & \overline{\partial}_{V_1}\end{pmatrix},$$

where $\beta\in\Omega^{0,1}(X;V_1\otimes V_0^*)$. Then define $Z:=\pi_X^*(V_0\oplus V_1)$ with the following $\overline{\partial}$-operator:

$$\overline{\partial}_Z|_{X\times\{t\}}=\begin{pmatrix}\overline{\partial}_{V'} & t\beta\\ 0 & \overline{\partial}_{V''}\end{pmatrix}.$$

This clearly satisfies the desired properties. □

6.2. **The functor μ and topology of vector bundles.** Here we will assume that X is connected. All the cohomology groups appearing in this section will be taken with coefficients in $\mathbb{R}$.

Let $Y=X\times\mathbb{P}^1$, and consider the action of $\mathbb{C}^\times$ on Y given by $\theta\cdot(z,[y:w])=(z,[y:\theta w])$.

Lemma 6.3. *There is a $\mathbb{C}^\times$-equivariant map $q:X_D\to Y$ which identifies X_D with the blow up of Y along $Z=D\times\{[0:1]\}$ and such that the exceptional divisor $E=q^{-1}Z$ corresponds to Δ^+.*

Proof. Recall that by definition $X_D \subset \mathbb{P}(L \oplus \underline{\mathbb{C}} \oplus \underline{\mathbb{C}})$ and that $p : X_D \to X$ denotes the projection. Let us identify $\mathbb{P}(\{0\} \oplus \underline{\mathbb{C}} \oplus \underline{\mathbb{C}})$ with $\mathbb{P}(\underline{\mathbb{C}} \oplus \underline{\mathbb{C}}) = X \times \mathbb{P}^1 = Y$ by mapping $[0 : y : w] \in \mathbb{P}(\{0\} \oplus \underline{\mathbb{C}} \oplus \underline{\mathbb{C}})_z$ to $(z, [y : w])$. This bijection is $\mathbb{C}^\times$-equivariant. With this in mind we will define the desired map q from X_D to $\mathbb{P}(\{0\} \oplus \underline{\mathbb{C}} \oplus \underline{\mathbb{C}})$.

Let $z \in X$ and let $v \in p^{-1}(z)$ be any point. Let $u = p^{-1}(z) \cap X_2$, and let $\Lambda_v \subset \mathbb{P}(L \oplus \underline{\mathbb{C}} \oplus \underline{\mathbb{C}})_z$ be the line passing through u and v if $u \neq v$, or the tangent of $p^{-1}(z)$ at u if $u = v$. Then we set

$$q(v) := \mathbb{P}(\{0\} \oplus \underline{\mathbb{C}} \oplus \underline{\mathbb{C}})_z \cap \Lambda_v.$$

Let us check that q is the blow up of Y along Z and that q is $\mathbb{C}^\times$-equivariant. For this it suffices to work locally in X, so that we may assume that $X = \mathbb{C}^n$ and $D = \{0\} \times \mathbb{C}^{n-1}$. Then $X_D = H_n$, and the map q can be described as follows. A point $v = [x : y : w] \in \{z\} \times \mathbb{P}(\mathbb{C}^3)$ satisfying $xy = w^2 t_1$, where $z = (t_1, \dots, t_n)$, is mapped to

$$q(v) = \begin{cases} (z, [y : w]) & \text{if } [x : y : w] \neq [1 : 0 : 0] \\ (z, [0 : 1]) & \text{if } [x : y : w] = [1 : 0 : 0] \end{cases}$$

From this it is clear that q is $\mathbb{C}^\times$-equivariant. It also follows that for any $(z, [y : w])$ such that $y \neq 0$ the preimage $r^{-1}(z, [0 : y : w])$ is equal to $\{(z, [y^{-1}w^2 t_1 : y : w])\}$, so that the restriction of q to $q^{-1}\{y \neq 0\}$ is an isomorphism. The blow up of $\{w \neq 0\}$ along Z (note that $Z \subset \{w \neq 0\}$) is by definition

$$B = \{(t_1, \dots, t_n, [y : 1], [a : b]) \mid a t_1 = b y\}.$$

It now suffices to construct a map $\Psi : B \to X_D$ which induces a biholomorphism between B and $\Psi(B)$ and which makes the following diagram commute:

$$\begin{tikzcd} B \arrow[rr, "\Psi"] \arrow[dr, "r"'] & & X_D \arrow[dl, "q"] \\ & Y, & \end{tikzcd}$$

where r is the composition of the blow up map $B \to \{w \neq 0\}$ with the inclusion $\{w \neq 0\} \subset Y$. The following definition suits our needs:

$$\Psi(t_1, \dots, t_n, [y : 1], [a : b]) = \begin{cases} (t_1, \dots, t_n, [t_1 : y^2 : y]) & \text{if } (t_1, y) \neq (0, 0), \\ (t_1, \dots, t_n, [b : 0 : a]) & \text{if } (t_1, y) = (0, 0). \end{cases}$$

From this explicit computation and (3.1) it follows that the exceptional divisor of q is Δ^+. ☐

Let N be the normal bundle of the inclusion $Z \subset Y$. Then $E = \mathbb{P}(N)$.

Lemma 6.4. *There is an isomorphism of cohomology with real coefficients*

$$H^2(X_D) = H^2(X) \oplus H^2(\mathbb{P}^1) \oplus \mathbb{R}\langle t \rangle,$$

where t is the first Chern class the line bundle associated to E.

Proof. This follows from Lemma 6.3 and a standard application of Mayer–Vietoris. ☐

Denote by $i : E \to X_D$ the inclusion of the exceptional divisor, and let $N_E \to E$ be the normal bundle. By a slight abuse of notation, if $a \in H^*(E)$, $t \cup a$ will denote the image of a by the composition

$$H^*(E) \to H^*(N_E, N_E \setminus \{0\}) \to H^*(X_D, X_D \setminus E) \to H^*(X_D), \tag{6.13}$$

where the first map is Thom's isomorphism, the second one is excission, and the third one is the map induced by the inclusion $(X_D, \emptyset) \subset (X_D, X_D \setminus E)$. Similarly, for any natural number $n \geq 1$, $t^n \cup a$ will sometimes denote $t \cup (i^* t^{n-1} \cup a)$, where the first $\cup$ is the one defined by (6.13) and the second one is the usual one in $H^*(E)$. For example, if $a = i^* b$ for some $b \in H^*(X_D)$, then

$$t^n \cup a = t^n \cup b, \tag{6.14}$$

where the $\cup$ in the RHS is the usual one in $H^*(X_D)$. Finally, we extend the definition by linearity in order to give a sense to expressions of the form $P(t) \cup a$, where P is a polynomial satisfying $P(0) = 0$.

Lemma 6.5. *Let $W \in \mathscr{V}_{\mathbb{C}^\times}(X_D)$, and let $(V, V_1) := \mu(W)$. The Chern character of W is*

$$\operatorname{ch} W = p^* \operatorname{ch} V + (e^t - 1) \cup p^* \operatorname{ch} V_1.$$

Proof. Take some $W \in \mathscr{V}_{\mathbb{C}^\times}(X_D)$ and set $(V, V_1) := \mu(W)$. Let also $W_0 \in \mathscr{V}_{\mathbb{C}^\times}(X_D)$ be such that $\mu(W_0) \simeq (V, 0)$. By Corollary 4.4, W_0 is isomorphic to $p^* V$ with the trivial action of $\mathbb{C}^\times$. So

$$\operatorname{ch} W_0 = p^* \operatorname{ch} V. \tag{6.15}$$

Let $X_D^* = X_D \setminus p^{-1} D$. Consider the cohomology long exact sequence for the pair (X_D, X_D^*):

$$\cdots \longrightarrow H^k(X_D, X_D^*) \xrightarrow{i} H^k(X_D) \xrightarrow{j} H^k(X_D^*) \longrightarrow \dots$$

By Corollary 4.4 the restrictions $W|_{X_D^*}$ and $W_0|_{X_D^*}$ are isomorphic. Hence $j(\operatorname{ch} W - \operatorname{ch} W_0) = 0$, which implies that $\operatorname{ch} W - \operatorname{ch} W_0 = i(d_{W,W_0})$ for some class $d_{W,W_0} \in H^*(X_D, X_D^*)$. On the other hand, if B is any neighbourhood of D, we have an isomorphism induced by excission

$$H^*(X_D, X_D^*) \simeq H^*(p^{-1} B, p^{-1}(B \setminus D)),$$

so that to compute $\operatorname{ch} W - \operatorname{ch} W_0$ it suffices to compute $\operatorname{ch} W|_{p^{-1}B} - \operatorname{ch} W_0|_{p^{-1}B}$ for any neighbourhood B of D.

Now, by Lemmae 6.1 and 6.2 we can reduce the computation to the case in which V splits as $V_a \oplus V_b$ and V_1 is equal to the restriction $V_a|_D$. Then, for any $W_a, W_b \in \mathscr{V}_{\mathbb{C}^\times}(X_D)$ such that $\mu(W_a)$ (resp. $\mu(W_b)$) is isomorphic to $(V_a, V_a|_D)$ (resp. $(V_b, 0)$) there is a C^∞ isomorphism $W|_B = W_a \oplus W_b$. By Corollary 4.4 we have $\operatorname{ch} W_b = p^* \operatorname{ch} V_b$. Let $\mathscr{L} \to X_D$ be the line bundle constructed in Subsection 4.4. One can check that $W_a' = W_a \otimes \mathscr{L}|_{p^{-1}B}^{-1}$ belongs to $\mathscr{V}_{\mathbb{C}^\times}(p^{-1}B)$ and that all its weights are zero. Hence, by Corollary 4.4, we have isomorphisms

$$W_a' \simeq p^* W_a'|_{X_2 \cap p^{-1}B} \simeq p^* W_a|_{X_2 \cap p^{-1}B}$$

(the last isomorphism follows from the existence of the nowhere zero section $\psi \in H^0(X_2;\mathscr{L}|_{X_2})$). So we can compute the Chern character: $\operatorname{ch} W_a = \operatorname{ch}\mathscr{L} \cup p^* \operatorname{ch} V_a$. But by construction the first Chern class of $\mathscr{L}$ is t, so $\operatorname{ch} W_a = e^t \cup p^* \operatorname{ch} V_a$ and consequently $\operatorname{ch} W|_{p^{-1}B} = p^* \operatorname{ch} V_b + e^t \cup p^* \operatorname{ch} V_a$. Hence,

$$\begin{aligned} \operatorname{ch} W|_{p^{-1}B} - \operatorname{ch} W_0|_{p^{-1}B} &= p^* \operatorname{ch} V_b + e^t \cup p^* \operatorname{ch} V_a - p^* \operatorname{ch}(V_a \oplus V_b) \\ &= (e^t - 1) \cup p^* \operatorname{ch} V_a \\ &= (e^t - 1) \cup p^* \operatorname{ch} V_1, \end{aligned} \tag{6.16}$$

where the last inequality follows from (6.14) and in the last line the cup product is the one defined by (6.13). Combining (6.15) with (6.16) the result follows. □

Corollary 6.6. *Let $W \in \mathscr{V}_{\mathbb{C}^\times}(X, D)$, and let $(V, V_1) = \mu(W)$. Then*

$$c_1(W) = p^* c_1(V) + \operatorname{rk} V_1 t.$$

6.3. **The Kaehler cone of certain blow ups.**

6.3.1. In order to compute the Kaehler cone of X_D we will prove a general result describing the Kaehler cone of the blow up of a product of compact Kaehler manifolds $X \times X'$ along $D \times D'$, where $D \subset X$ and $D' \subset X'$ are smooth divisors. This is done in this subsection, the main result being stated in Theorem 6.8.

Let N be a manifold and $\pi : \Lambda \to N$ a Hermitian line bundle, and assume that a compact connected Lie group T acts on Λ on the left, linearly on the fibres and respecting the metric. Let $\mathfrak{t} = \operatorname{Lie} T$. For any $s \in \mathfrak{t}$ denote by $\mathfrak{X}_\Lambda(s)$ the vector field generated by the infinitesimal action of s on Λ and by $\mathfrak{X}_N(s) = d\pi \mathfrak{X}_\Lambda(s)$ the vector field generated on N.

Fix a T-invariant Hermitian connection B on Λ. This defines for any vector field $\mathfrak{X} \in \Gamma(TN)$ a lift $\sigma_B(\mathfrak{X}) \in \Gamma(T\Lambda)$. We define the moment of the action of T on Λ w.r.t. B to be the map $\Omega_B : \mathfrak{t} \to \Omega^0(N; \sqrt{-1}\mathbb{R})$ which assigns to $s \in \mathfrak{t}$ the field

$$\Omega_B(s) := \sigma_B(\mathfrak{X}_N(s)) - \mathfrak{X}_\Lambda(s).$$

This is easily seen to be a vertical vector field and, using the canonical identification $T\Lambda_v \simeq \pi^*\Lambda$, it is in fact a Hermitian linear map, so we can view $\Omega_B(s) \in \Omega^0(N; \sqrt{-1}\mathbb{R})$. A straightforward computation proves that for any field $\mathfrak{X} \in \Gamma(TN)$ we have

$$d\Omega_B(s) = i(\mathfrak{X}_N(s)) F_B, \tag{6.17}$$

where $F_B \in \Omega^2(N; \sqrt{-1}\mathbb{R})$ is the curvature of B.

Now assume that M is another manifold, and that $P \to M$ is a T-principal bundle. We will construct a connection ∇ on the line bundle

$$\pi_M : P \times_T \Lambda \to P \times_T N,$$

and for that it suffices to specify a suitable retraction $\nu_\nabla : T(P \times_T \Lambda) \to \operatorname{Ker} d\pi_M$. Let $p : P \times_T \Lambda \to M$ be the projection. Take a connection A on P. Then we get a retraction

$\nu_A : T(P \times_T \Lambda) \to \operatorname{Ker} dp = P \times_T T\Lambda$. On the other hand, the connection B gives a T-equivariant retraction $\nu_B : T\Lambda \to \operatorname{Ker} d\pi$ which consequently extends to a retraction

$$\nu'_B : P \times_T T\Lambda \to P \times_T \operatorname{Ker} d\pi = \operatorname{Ker} d\pi_M.$$

We then define

$$\nu_\nabla := \nu'_B \circ \nu_A.$$

It is easy to see that this defines a connection ∇ on $P \times_T \Lambda \to P \times_T N$.

Lemma 6.7. *The curvature of ∇ is*

$$F_\nabla = {\nu'_A}^* F_B - \Omega_B(q^* F_A),$$

where $q : P \times_T N \to M$ is the projection and $\nu'_A : T(P \times_T N) \to \operatorname{Ker} dq$ is the retraction induced by A.

Proof. This follows from an easy local computation (see for example Lemma A.1.4 in [M]). Note, by the way, that F_∇ is the coupling form of the bundle $P \times_T N$, see [GLS]. □

6.3.2. Let $N = \mathbb{P}^1$ with the action of $T = S^1 \times S^1$ given by

$$(\theta_1, \theta_2)[x : y] = [\theta_1 x : \theta_2 y]. \tag{6.18}$$

This action lifts to $\Lambda := \mathscr{O}(-1) \to \mathbb{P}^1$ by defining $(\theta_1, \theta_2)(\lambda x, \lambda y) = (\theta_1 \lambda x, \theta_2 \lambda y)$, where we identify the fibre of $\mathscr{O}(-1)$ over $[x : y]$ with $\{(\lambda x, \lambda y) \mid \lambda \in \mathbb{C}\}$. Take the T-invariant metric $|(\lambda x, \lambda y)|^2 = |\lambda x|^2 + |\lambda y|^2$ on Λ, and let B be its Chern connection. Then B is T-invariant, and its curvature satisfies

$$F_B = 2\pi\sqrt{-1}\omega_{\mathbb{P}^1},$$

where $\omega_{\mathbb{P}^1}$ is the Fubini-Study symplectic form on $\mathbb{P}^1$. Identifying $\mathfrak{t} = \operatorname{Lie} T \simeq \sqrt{-1}\mathbb{R} \oplus \sqrt{-1}\mathbb{R}$, let $t_1 = (i, 0) \in \mathfrak{t}$, $t_2 = (0, i) \in \mathfrak{t}$, and let τ_1, τ_2 be the dual basis of t_1, t_2. Define

$$\mu([x : y]) := \frac{\tau_1 |x|^2 + \tau_2 |y|^2}{|x|^2 + |y|^2} \in \mathfrak{t}^*.$$

Then μ is a symplectic moment map for the action of T on $\mathbb{P}^1$ with respect to $\omega_{\mathbb{P}^1}$. It follows from (6.17) that $\Omega_B = \mu + C$ for some constant $C \in \mathfrak{t}^*$. Let us compute this constant. The subgroup $G = S^1 \times \{1\} \subset T$ acts trivially on the fibre $\Lambda_{[0:1]}$ and the action on the tangent space $T_{[0:1]}\mathbb{P}^1$ has weight 1. So for any $x \in \Lambda_{[0:1]}$ the tangent space $T_x\Lambda$ splits by the action of G in weights 0 and 1. And, since the connection B is G-invariant, the horizontal lift of $T_{[0:1]}\mathbb{P}^1$ must be the inclusion in the summand of weight 1 in $T_x\Lambda$. But the value of the vector field $\mathfrak{X}_\Lambda(t_1)$ at x lies also inside the same summand. Consequently, $\Omega(t_1)([0 : 1]) = 0$. Similarly one checks that $\Omega(t_2)([1 : 0]) = 0$, so we deduce that $C = 0$, i.e., $\Omega_B = \mu$.

6.3.3. For any manifold Z we will denote by $K(Z) \subset H^{1,1}(Z;\mathbb{C}) \cap H^2(Z;\mathbb{R})$ the Kaehler cone of Z.

Theorem 6.8. *Let X, X' be Kaehler manifolds, with X' simply connected, and let $D \subset X$, $D' \subset X'$ be smooth divisors. Let $c = c_1(D)$ and $c' = c_1(D')$. Let Y be the blow up of $X \times X'$ at $D \times D'$. Then we have*

$$H^2(Y;\mathbb{R}) = H^2(X;\mathbb{R}) \oplus H^2(X';\mathbb{R}) \oplus \mathbb{R}\langle t\rangle, \tag{6.19}$$

where t is the first Chern class of the exceptional divisor. The Kaehler cone of Y is

$$K(Y) = \left\{ (w, w', bt) \in H^2(Y;\mathbb{R}) \;\middle|\; \begin{array}{l} w \in K(X),\ w + bc \in K(X), \\ w' \in K(X'),\ w' + bc' \in K(X'), \\ \textit{and } b < 0. \end{array} \right\}.$$

Suppose now that S^1 acts on X' keeping D' fixed, so that there is an induced action of S^1 on $X \times X'$ and the blow up Y. Then any class in $K(Y)$ can be represented by a S^1-invariant Kaehler form.

Proof. Formula (6.19) follows from Künneth (using $H^1(X';\mathbb{R}) = 0$) and an easy Mayer–Vietoris argument, exactly as in Lemma 6.4. Let $L \to X$ and $L' \to X'$ be the line bundles defined by D and D'. Since the normal bundle $N_{D\times D'|X\times X'} = L|_D \boxplus L'|_{D'}$, there is an embedding

$$i_Y : Y \to \mathbb{P}(L \boxplus L') =: Q.$$

More precisely, let $\sigma \in H^0(L)$ and $\sigma' \in H^0(L')$ be sections transverse to zero such that $\sigma^{-1}(0) = D$ and ${\sigma'}^{-1}(0) = D'$. Then $i_Y(Y)$ coincides with the closure in Q of the image of the section

$$\begin{array}{rcl} X \times X' \setminus D \times D' & \longrightarrow & Q \\ (x, x') & \mapsto & [\sigma(x) : \sigma'(x')]. \end{array}$$

Let $x_0 \in D$ and $x_0' \in D'$. Let $i_0 : X \setminus D \to Q|_{X\times\{x_0'\}}$ and $i_0' : X' \setminus D' \to Q|_{\{x_0\}\times X'}$ be the maps $i_0(x) = [\sigma(x) : 0]$ and $i_0'(x') = [0 : \sigma'(x')]$. Both i_0 and i_0' extend to give embeddings $i : X \to Y$ and $i' : X' \to Y'$ satisfying

$$i^*(w, w', bt) = w + bc \qquad \text{and} \qquad i'^*(w, w', bt) = w' + bc'.$$

Taking now $x_0 \in X \setminus D$ and $x_0' \in X' \setminus D'$ and repeating the same construction we get embeddings $j : X \to Y$ and $j' : X' \to Y$ such that $\operatorname{Im} i_Y j \subset Q|_{X\times\{x_0'\}}$ and $\operatorname{Im} i_Y j' \subset Q|_{\{x_0\}\times X'}$ and satisfying

$$j^*(w, w', bt) = w \qquad \text{and} \qquad j'^*(w, w', bt) = w'.$$

Now suppose that $(w, w', bt) \in K(Y)$. Then, since i, i', j, j' are embeddings, $w \in K(X)$, $w + bc \in K(X)$, $w' \in K(X')$ and $w' + bc' \in K(X')$ (indeed, the pullback by an embedding of a Kaehler form is a Kaehler form). Let $\pi : Y \to X \times X'$ denote the projection. To check that $b < 0$, take any point $z = (x, x') \in D \times D'$. Then $Z := \pi^{-1}(z) = \mathbb{P}(L_x \oplus L'_{x'})$ and the restriction of t to Z is $t|_Z = c_1(\mathscr{O}_{\mathbb{P}(L_x \oplus L'_{x'})}(-1))$. Hence,

$$\int_Z w + w' + bt = \int_Z bt = -b,$$

and this must be positive if (w, w', bt) is a Kaehler class.

Conversely, let $(w, w', bt) \in H^2(Y;\mathbb{R})$ satisfy $w \in K(X)$, $w + bc \in K(X)$, $w' \in K(X')$, $w' + bc' \in K(X')$ and $b < 0$. We will construct a Kaehler form $\kappa \in \Omega^2(Y)$ representing the class $w+w'+bt$. Let $\omega_0, \omega_1 \in \Omega^2(X)$ and $\omega'_0, \omega'_1 \in \Omega^2(X')$ be Kaehler forms satisfying $[\omega_0] = w$, $[\omega_1] = w + bc$, $[\omega'_0] = w'$ and $[\omega'_1] = w' + bc'$. Define $F = -2\pi\sqrt{-1}b^{-1}(\omega_1 - \omega_0)$ and $F' = -2\pi\sqrt{-1}b^{-1}(\omega'_1 - \omega'_0)$.

Fix metrics on L and L'. One can then take Hermitian connections C on L and C' on L' such that the curvatures satisfy

$$F_C = F \qquad \text{and} \qquad F_{C'} = F'.$$

Let $P_L \subset L$ and $P_{L'} \subset L'$ be the unit length vectors. Both P_L and $P_{L'}$ are S^1 principal bundles. Define $M := X \times X'$ and $P = \pi_X^* P_L \times \pi_{X'}^* P_{L'}$. Then P is a $T := S^1 \times S^1$ principal bundle.

Using the action of T on $\mathbb{P}^1$ defined by (6.18), we have $P \times_T \mathbb{P}^1 = Q$. By the Künneth and Leray–Hirsch theorems and the fact that $H^1(\mathbb{P}^1;\mathbb{R}) = 0$ we have

$$H^2(Q;\mathbb{R}) = H^2(X;\mathbb{R}) \oplus H^2(X';\mathbb{R}) \oplus \mathbb{R}\langle t_Q\rangle,$$

where $t_Q = c_1(\mathscr{O}_Q(-1))$ is the relative canonical bundle. Taking the bundle $\Lambda = \mathscr{O}_{\mathbb{P}^1}(-1)$ and the lift of the action of T defined in 6.18, we also have

$$\mathscr{O}_Q(-1) = P \times_T \Lambda.$$

Let A be the connection on P induced by the pullbacks of the connections C and C'. Then $F_A = F + F' \in \Omega^2(X \times X')$ (we omit the pullbacks). Taking on Λ the connection specified in 6.18, we can now apply the construction in 6.3.1 to get a connection ∇ on $\mathscr{O}_Q(-1)$. Then we define

$$\kappa_Q := \frac{\sqrt{-1}}{2\pi}(\pi^*(\omega_0 + \omega'_0) + bF_\nabla).$$

It is clear that this form represents $[\kappa_Q] = (w, w', bt_Q)$. Let us check that it is a Kaehler form, i.e., that it is positive. Take a point $y \in Q$ and a nonzero $0 \neq v \in T_yQ$. We want to prove that

$$\kappa_Q(v, iv) > 0.$$

Denote by $\pi_Q : Q \to X \times X'$ the projection, and write $\pi_Q(y) = (x, x')$ and $d\pi_Q(v) = (u, u') \in T_xX \oplus T_{x'}X'$. Let also $v_0 = \nu'_A(v)$ be the vertical projection of v. Suppose that $y = [y_0 : y_1] \in \mathbb{P}(L_x \oplus L'_{x'})$. By Lemma 6.7 and (6.19) we have

$$\begin{aligned}\kappa_Q(v, iv) = \frac{\sqrt{-1}}{2\pi}&\left(bF_B(v_0, iv_0) + b\frac{|y_0|^2}{|y_0|^2 + |y_1|^2}F(u, iu) + b\frac{|y_1|^2}{|y_0|^2 + |y_1|^2}F'(u', iu')\right)\\ &+ \omega_0(u, iu) + \omega'_0(u', iu').\end{aligned}$$

By construction, $\frac{\sqrt{-1}}{2\pi}bF_B(v_0, iv_0) > 0$. As for the other terms, recall that

$$\frac{\sqrt{-1}b}{2\pi}F = \omega_1 - \omega_0 \qquad \text{and} \qquad \frac{\sqrt{-1}b}{2\pi}F' = \omega'_1 - \omega'_0,$$

so the remaining summands in $\kappa_Q(v, iv)$ can be written as

$$(\lambda\omega_0 + (1-\lambda)\omega_1)(u, iu) + ((1-\lambda)\omega'_0 + \lambda\omega'_1)(u', iu'),$$

for some $0 \le \lambda \le 1$, and this is positive since by assumption $\omega_0, \omega_1, \omega'_0, \omega'_1$ are Kaehler forms.

Finally, we set $\kappa := i_Y^* \kappa_Q$. Since i_Y is an embedding, κ is a Kaehler form, and since $i_Y^* t_Q = t$, it follows that κ represents the class (w, w', bt) as desired.

It remains to prove the last statement on existence of S^1-invariant Kaehler forms. This follows from the standard averaging trick. $\square$

6.4. **The Kaehler cone of $X(D, r)$.** In the rest of the present section we will assume that X is Kaehler. Recall that $X(D, r)$ fits in a tower of maps

$$X(D, r) = Y_r \to Y_{r-1} \to \cdots \to Y_0 = X.$$

We will describe the Kaehler cone of Y_j for any $1 \le j \le r$. We first define (using induction on j) cohomology classes

$$d_{j,1}, \ldots, d_{j,j}, t_{j,1}, \ldots, t_{j,j}$$

in $H^2(Y_j)$. Recall that $Y_1 = X_D$ is isomorphic to the blow up of $Y = X \times \mathbb{P}^1$ along $D \times \{[0:1]\}$ (see Lemma 6.3). Let $q : X_D \to Y$ be the blow up map, and let $\delta = 1 \otimes PD[\mathbb{P}^1] \in H^0(X) \otimes H^2(\mathbb{P}^1) \subset H^2(Y)$. Then we set $d_{1,1} := q^*\delta$, and we define $t_{1,1}$ to be first Chern class of the exceptional divisor $\Delta_1^+ = q^{-1}(D \times \{[0:1]\}) \subset Y_1$. Now assume that $1 \le j < r$ and that the classes $d_{j,1}, \ldots, d_{j,j}, t_{j,1}, \ldots, t_{j,j} \in H^2(Y_j)$ have already been defined. Recall that $Y_{j+1} = (Y_j)_{\Delta_j^+}$. Using again Lemma 6.3 we can factor the projection map $p_{j+1} : Y_{j+1} \to Y_j$ as follows

$$\begin{tikzcd} Y_{j+1} \arrow[dd, "p_{j+1}"'] \arrow[dr, "q_{j+1}"] & \\ & Y_j \times \mathbb{P}^1 \arrow[dl, "o_j"] \\ Y_j, & \end{tikzcd}$$

where q_{j+1} is the blow up map of $Y_j \times \mathbb{P}^1$ along $\Delta_j^+ \times \{[0:1]\}$. We then define for any $1 \le i \le j$

$$d_{j+1,i} := p_{j+1}^* d_{j,i} \qquad \text{and} \qquad t_{j+1,i} := p_{j+1}^* t_{j,i}.$$

We also set $d_{j+1,j+1} := q_{j+1}^*(1 \otimes PD[\mathbb{P}^1])$ and $t_{j+1,j+1} = c_1(\Delta_{j+1}^+)$. In the sequel, by an abuse of notation, we will omit the first subindex in the variables $d_{j,i}, t_{j,i}$, so that d_i and t_i will denote cohomology classes in Y_j for every $j \ge i$).

For any $1 \le j \le r$ let ρ_j denote the composition $p_j \circ \cdots \circ p_1 : Y_j \to X$.

Lemma 6.9. *Let $1 \le j \le r$. The map $\rho_j^* : H^2(X) \to H^2(Y_j)$ is injective and, identifying $H^2(X)$ with its image in $H^2(Y_j)$ by this map, we have*

$$H^2(Y_j) = H^2(X) \oplus \mathbb{R}\langle d_1, \dots, d_j, t_1, \dots, t_j \rangle.$$

Proof. It follows from applying recursively Lemma 6.4. □

Lemma 6.10. *Let $1 \le j \le r$. The map $\rho_j^* : H^2(X) \to H^2(Y_j)$ is injective and, identifying $H^2(X)$ with its image in $H^2(Y_j)$ by this map, we have*

$$H^2(Y_j) = H^2(X) \oplus \mathbb{R}\langle d_1, \dots, d_j, t_1, \dots, t_j \rangle.$$

Let $c = c_1(D) \in H^2(X)$. The Kaehler cone of Y_j is equal to

$$K(Y_j) = \left\{ (\omega, a, b) \in H^2(Y_j) \;\middle|\; \begin{array}{ll} & \text{satisfying, for any } 1 \le i \le j, \\ a \in \mathbb{R}^j,\ b \in \mathbb{R}^j, & a_i > 0,\ b_i < 0, \\ \omega \in K(X), & \omega + (b_1 + \dots + b_i)c \in K(X), \\ & a_i + b_i + b_{i+1} + \dots + b_j > 0. \end{array} \right\},$$

where for any $\omega \in H^2(X)$, $a = (a_1, \dots, a_j) \in \mathbb{R}^j$ and $b = (b_1, \dots, b_j) \in \mathbb{R}^j$ the triple (w, a, b) denotes

$$(\omega, a, b) := \omega + \sum_{i=1}^{j} a_i d_i + b_i t_i \in H^2(Y_j).$$

Proof. Use induction on j and apply at each step Theorem 6.8 with $X' = \mathbb{P}^1$. □

7. Slope of vector bundles (smooth divisor)

The aim of this section is to study the behaviour of the slope of vector bundles under the functor $\mu(r)$. We will throughout assume that X is Kaehler and compact. We will use the notations of the preceding sections, specially 6.4 (recall, by the way, that the notion of slope is intrinsically related to the Kaehler cone of the base manifold — see (7.23)). The main result will be stated in Theorem 7.5. Before, we state and prove some technical lemmae.

Denote by $\partial_l : \Delta_l^+ \hookrightarrow Y_l$ the inclusion of the exceptional divisor, and let $m_l : \Delta_l^+ \to \Delta_{l-1}^+$ be the restriction of the projection p_l. We then have a chain of maps

$$\Delta_r^+ \xrightarrow{m_r} \Delta_{r-1}^+ \xrightarrow{m_{r-1}} \cdots \xrightarrow{m_1} \Delta_0^+ = D. \tag{7.20}$$

We now list some properties of cohomology classes defined in 7.5.

Lemma 7.1. *(1) $d_l^2 = 0$ and $\partial_l^* d_l = 0$;*

(2) for any $\alpha \in H^*(Y_l)$, $\langle t_l \alpha, [Y_l] \rangle = \langle \partial_l^* \alpha, [\Delta_l^+] \rangle$;

(3) $t_l^2 = -t_l t_{l-1}$ inside $H^*(\Delta_l^+)$; for any $\alpha \in H^*(\Delta_{l-1}^+)$,

$$\langle t_l m_l^* \alpha, [\Delta_l^+] \rangle = -\langle \alpha, [\Delta_{l-1}^+] \rangle,$$

and $\langle m_l^*\alpha, [\Delta_l^+]\rangle = 0$*;*

(4) suppose that $\alpha \in H^*(Y_l)$ *and that* $\langle \alpha, [Y_l]\rangle \neq 0$ *and* α *is not of the form* $t_l\alpha'$ *for some* $\alpha' \in H^*(Y_l)$*, then* $\alpha = d_l p_l^*\alpha''$ *for some* $\alpha'' \in H^*(Y_{l-1})$*, and* $\langle \alpha, [Y_l]\rangle = \langle \alpha'', [Y_{l-1}]\rangle$*.*

Proof. (1) The first claim is obvious, and the second one follows from the commutativity of the diagram

$$\begin{tikzcd} \Delta_l^+ \arrow[r, "\partial_l"] \arrow[d, "m_l"'] & Y_l \arrow[d, "q_l"] \\ \Delta_{l-1}^+ \times \{[0:1]\} \arrow[r, "n_l"] & Y_{l-1} \times \mathbb{P}^1, \end{tikzcd}$$

and the fact that $n_l^* PD(1\otimes[\mathbb{P}^1]) = 0$. (2) This holds because t_l is the Poincaré dual of the fundamental class $[\Delta_l^+]$. (3) Since Δ_l^+ is the exceptional divisor of the blow up of $Y_{l-1}\times\mathbb{P}^1$ along $\Delta_{l-1}^+\times\{[0:1]\}$, we can identify $\Delta_l^+ = \mathbb{P}(N_{l-1}^+\oplus\underline{\mathbb{C}})$ (recall that $N_{l-1}^+ \to \Delta_{l-1}^+ \subset Y_{l-1}$ is the normal bundle), and the map m_l is the projection $\mathbb{P}(N_{l-1}^+\oplus\underline{\mathbb{C}}) \to \Delta_{l-1}^+$; with this in mind, we apply Leray's theorem together with the fact that $c_1(N_{l-1}^+) = t_{l-1}$. (4) If α cannot be written $\alpha = t_l\alpha'$, then certainly $\alpha = q_l^*\beta$ for some $\beta \in H^*(Y_{l-1}\times\mathbb{P}^1)$; furthermore, since the map q_l is birational, we have $\langle \alpha, Y_l\rangle = \langle \beta, Y_{l-1}\times\mathbb{P}^1\rangle$. Now the claim follows from Künneth's theorem. □

For any $I = (i_1,\dots,i_r) \in \mathbb{Z}_{\geq 0}^r$ we define

$$|I| = \sum_{l=1}^r i_l \qquad \text{and} \qquad \sigma(I) = \sum_{l=1}^r i_l 2^{l-1}.$$

We will use the standard multiindex notation, so that for example d^I will denote $\prod_{l=1}^r d_l^{i_l}$.

Lemma 7.2. *Suppose that for some* $I = (i_1,\dots,i_r), J = (j_1,\dots,j_r) \in \mathbb{Z}_{\geq 0}^r$ *and* $\eta \in H^*(X)$ *we have* $\langle d^I t^J \eta, [X(D,r)]\rangle \neq 0$*. Then there exists some* $0 \leq l \leq r$ *such that*

(a) if $l < r$ *then* $I = (0,\dots,0,1,\dots,1)$ *(1's in the last* $r-l = |I|$ *positions and 0's everywhere else), and if* $l = r$ *then* $I = (0,\dots,0)$*; besides, for any* $l < \mu \leq r$*,* $i_\mu = 0$*;*

(b) for any $1 \leq \mu \leq l$ *we have* $\sum_{\mu\leq k\leq l} j_k \geq 2 + (l-\mu)$*;*

(c) we have $|I| + |J| \geq r$ *and* $\sigma(I) + \sigma(J) \geq 2^r - 1$*, with equality in any of the two if and only if* $J = 0$ *and* $I = (1,\dots,1)$ *(hence, the case* $l = r$*); in this situation*

$$\langle d^I t^J \eta, [X(D,r)]\rangle = \langle d^I \eta, [X(D,r)]\rangle = \langle \eta, [X]\rangle. \tag{7.21}$$

Proof. Throughout the proof we will refer to statements (1)-(4) in Proposition 7.1. Applying (1) and (4) from $l = r$ downwards as many times as possible (i.e., as long as we don't find some nonzero i_l or we arrive at $l = 0$), we deduce that I ends with a sequence of 1's of length $r-l$, where $0 \leq l \leq r$. Furthermore, the last $r-l$ positions of J vanish. Now suppose that $l \geq 0$. Then $i_l \neq 0$, and defining $I' = (i_1',\dots,i_r') \in \mathbb{Z}_{\geq 0}^r$ by $i_j' = i_j - \delta_{lj}$ we deduce from (4) and (2) that

$$\langle d^J t^I \eta, [X(D,r)]\rangle = \langle t^I \eta, [Y_j]\rangle = \langle t^{I'}, [\Delta_l^+].$$

Then (1) tells us that the first l positions of I have to vanish, so we are done with (a). We now apply successively (3) to descend step by step the tower (7.20), until we arrive at D, and we prove (b). We now prove (c). First of all, it is clear that if $J = 0$ and $I = (1, \ldots, 1)$ then $\sigma(I) + \sigma(J) = 2^{r-1} - 1$. If this is not the case then $1 \leq l \leq r$, and J satisfies (b). It follows from this that

$$\sigma(J) = \sum_{1 \leq k \leq l} j_k 2^{k-1} > 2^l.$$

Let us see why. If $J = J_0 := (1, 1, \ldots, 1, 2, 0, \ldots, 0)$ then we are done. Now, no matter what J is, we can modify it by a sequence of moves in such a way that (b) is preserved all the time, $\sigma(J)$ does not increase, and at the end we arrive at J_0. Indeed, by (c) we have in general $j_l \geq 2$. If $j_l > 2$, then we shift $j_l - 2$ units from position l to position $l - 1$. This move preserves (c) and makes $\sigma(J)$ decrease. Next we look at j_{l-1}, which is ≥ 1 by (c). If $j_{l-1} > 1$, then we shift $j_{l-1} - 1$ units from position $l - 1$ to position $l - 2$. And so on, until $J = (a, 1, \ldots, 1, 2, 0, \ldots, 0)$, where $a \geq 1$. Then we substitute a by 1, and we are done. $\square$

Lemma 7.3. *Suppose that $I = (i_1, \ldots, i_r), J = (j_1, \ldots, j_r) \in \mathbb{Z}^r_{\geq 0}$ and $\eta \in H^*(X)$. If for some $1 \leq l \leq r$ we have $\langle t_l d^I t^J \eta, [X(D, r)] \rangle \neq 0$ then*

$$|I| + |J| \geq r \qquad \text{and} \qquad \sigma(I) + \sigma(J) \geq 2^r - 1,$$

with equality in any of the two if and only if there is some $1 \leq l \leq r$ such that $J = (1, \ldots, 1, 0, \ldots, 0)$ (1's in the first $l = |J|$ positions) and $I = (0, \ldots, 0, 1, \ldots, 1)$ (1's in the last $r - l$ positions). In this case we have

$$\langle t_l d^I t^J \eta, [X(D, r)] \rangle = (-1)^l \langle \eta, [D] \rangle. \tag{7.22}$$

Proof. This is completely analogous to the preceding lemma. (Formula (7.22) follows from applying recursively (3) in Lemma 7.1.) $\square$

Lemma 7.4. *For any sequence $\mathcal{V} = (V_1 \subset \cdots \subset V_r)$ of vector bundles and $1 \leq l \leq r$, we define $R_l(\mathcal{V}) := \operatorname{rk} V_l$. Let $W \in \mathscr{V}_{G(r)}(X(D, r))$, and define $(V, \mathcal{V}) := \mu(r)(W) \in \mathscr{P}(X, D, r)$. Then*

$$c_1(W) = c_1(V) + \sum_{1 \leq l \leq r} R_l(\mathcal{V}) t_l.$$

Proof. Use induction on j and apply at each step Corollary 6.6. $\square$

Assume that $\Omega := (\omega, a, b) \in H^2(X(D, r))$ belongs to the Kaehler cone of $X(D, r)$. Let $W \in \mathscr{V}_{G(r)}(X(D, r))$ be an equivariant vector bundle. The Ω-slope of W is by definition

$$\operatorname{slope}_\Omega W := \frac{1}{\operatorname{rk} W} \left\langle c_1(W) \frac{\Omega^{n+r-1}}{(n + r - 1)!}, [X(D, r)] \right\rangle. \tag{7.23}$$

(Recall that X has dimension n, so that the dimension of $X(D, r)$ is $n + r$.)

Fix a Kaehler class $\omega \in K(X)$, and let $\underline{\lambda} = (\lambda_1, \dots, \lambda_r)$, where

$$1 > \lambda_1 > \cdots > \lambda_r > 0$$

are real numbers (the parabolic weights). The $(\omega, \underline{\lambda})$-parabolic slope of $(V, \mathcal{V})$ is by definition

$$\text{par-slope}_{\omega,\underline{\lambda}}(V, \mathcal{V}) = \frac{1}{\operatorname{rk} V}\left(\left\langle c_1(V)\frac{\omega^{n-1}}{(n-1)!}, [X]\right\rangle + \sum_{i=1}^{r} R_i(\mathcal{V})\lambda_i \left\langle \frac{\omega^{n-1}}{(n-1)!}, [D]\right\rangle\right).$$

Define $\beta_1 := \lambda_1$ and, for any $1 < j \leq r$, $\beta_j := \lambda_j/\lambda_{j-1}$. For any $\epsilon \in \mathbb{R}$, let

$$\Omega(\omega, \underline{\lambda}, \epsilon) := \omega + \sum_{j=1}^{r} \epsilon^{2^{j-1}}(d_j - \beta_j t_j).$$

Theorem 7.5. *Let $W \in \mathscr{V}_{G(r)}(X(D, r))$, and let $(V, \mathcal{V}) := \mu(r)(W) \in \mathscr{P}(X, D, r)$.*

(1) If $\epsilon > 0$ is small enough, then $\Omega(\omega, \underline{\lambda}, \epsilon) \in K(X(D, r))$;

(2) $\operatorname{slope}_{\Omega(\omega,\underline{\lambda},\epsilon)} W$ is a polynomial in ϵ of the following form:

$$\sum_{\substack{J=(j,j')\\ j+j'=n+r}} \theta_J(\epsilon)\langle \omega^j [D]^{j'}, [X]\rangle + \sum_{\substack{J=(j,j')\\ j+j'=n+r-1}} \theta'_J(\epsilon)\langle c_1(V)\omega^j [D]^{j'}, [X]\rangle,$$

where both sums run over pairs of nonnegative integers, and the θ_J, θ'_J are polynomials in ϵ whose coefficients only depend on X, ω, $\underline{r}$, and Λ;

(3) We then have:

$$\operatorname{slope}_{\Omega(\omega,\underline{\lambda},\epsilon)}(W) = \epsilon^{2^r - 1}\,\text{par-slope}_{\omega,\underline{\lambda}}(V, \mathcal{V}) + O(\epsilon^{2^r}).$$

Proof. Statement (1) follows from Lemma 6.10. (2) follows from applying repeatedly Lemma 7.1. Let us prove statement (3). To save on typing, we will write R_l instead of $R_l(\mathcal{V})$. Combining Lemma 7.4 with the definition of $\Omega(\omega, \underline{\lambda}, \epsilon)$ we can write and develope

$$\begin{aligned}
\operatorname{slope}_{\Omega(\omega,\underline{\lambda},\epsilon)} W &= \frac{1}{\operatorname{rk} W}\left\langle \left(c_1(V) + \sum_{1\leq l\leq r} R_l t_l\right)\frac{\left(\omega + \sum_{j=1}^{r}\epsilon^{2^{j-1}}(d_j - \beta_j t_j)\right)^{n+r-1}}{(n+r-1)!}, [X(D, r)]\right\rangle \\
&= \frac{1}{\operatorname{rk} W}\sum_{k\in\mathbb{Z}_{\geq 0}, I,J\in\mathbb{Z}^r_{\geq 0}} \theta_{k,I,J}\left\langle c_1(V)\omega^k d^I(-\beta t)^J \epsilon^{\sigma(I)+\sigma(J)}, [X(D, r)]\right\rangle + \\
&+ \frac{1}{\operatorname{rk} W}\sum_{1\leq l\leq r}\sum_{k\in\mathbb{Z}_{\geq 0}, I,J\in\mathbb{Z}^r_{\geq 0}} \theta_{l,k,I,J} R_l\left\langle t_l\omega^k d^I(-\beta t)^J \epsilon^{\sigma(I)+\sigma(J)}, [X(D, r)]\right\rangle,
\end{aligned}$$

where the $\theta_{k,I,J}$ and $\theta_{l,k,I,J}$ are real numbers (note that k, I, J satisfy the relation $k + |I| + |J| = n + r - 1$). We now reduce mod ϵ^{2^r}. By Lemmae 7.2 and 7.3 the only terms which can possibly be nonzero are:

1. in the first summation, the one with $J = 0$ and $I = (1, \dots, 1)$, so $k = n-1$; then $\theta_{k,I,J} = \binom{n+r-1}{r} \frac{r!}{(n+r-1)!} = \frac{1}{(n-1)!}$;
2. in the second summation, the ones in which $J = (1, \dots, 1, 0, \dots, 0)$ (where 1 appears $l = |J|$ times, $1 \le l \le r$) and $I = (0, \dots, 0, 1, \dots, 1)$ (here 0 appears l times), so again $k = n-1$; then $\theta_{l,k,I,J} = \binom{n+r-1}{r} \binom{r}{i} \frac{i!(r-i)!}{(n+r-1)!} = \frac{1}{(n-1)!}$.

Taking this into account we now can write

$$\begin{aligned}\operatorname{slope}_{\Omega(\omega,\underline{\lambda},\epsilon)} W \equiv \frac{\epsilon^{2^r-1}}{\operatorname{rk} W} \Bigg(\Bigg\langle c_1(V) \prod_{1 \le j \le r} d_j \frac{\omega^{n-1}}{(n-1)!}, [X(D,r)] \Bigg\rangle + \\ + \sum_{i=1}^{r} R_i \Bigg\langle t_i \prod_{1 \le j \le i} (-\beta_j t_j) \prod_{i < j \le r} d_j \frac{\omega^{n-1}}{(n-1)!}, [X(D,r)] \Bigg\rangle \Bigg) \qquad \mod \epsilon^{2^r}.\end{aligned}$$

Finally, plugging in formulae (7.21) and (7.22) and taking into account that, for any i, $\prod_{1 \le j \le i} \beta_j = \lambda_i$, we deduce that

$$\operatorname{slope}_{\Omega(\omega,\underline{\lambda},\epsilon)} W \equiv \epsilon^{2^r-1} \operatorname{par-slope}_{\omega,\underline{\lambda}}(V, \mathcal{V}) \qquad \mod \epsilon^{2^r}.$$

This is what we wanted to prove. □

8. Parabolic structures over a normal crossing divisor

8.1. **The categories.** Let X be a manifold, and let $D \subset$ be a divisor with normal crossings. Assume that the irreducible components $D_1, \dots, D_s$ of D are smooth. Let us fix an s-tuple of nonzero natural numbers $\underline{r} = (r_1, \dots, r_s)$. Let $\mathscr{P}(X, D, \underline{r})$ be the category defined as follows:

1. The objects of $\mathscr{P}(X, D, \underline{r})$ are sequences $(V, \mathcal{V}_1, \dots, \mathcal{V}_s)$, where V is a vector bundle over X and where, for any $1 \le i \le s$, $\mathcal{V}_i$ denotes an increasing filtration of $V|_{D_i}$ of length r_i:
$$\mathcal{V}_i = (0 \subset V_{i,1} \subset \dots \subset V_{i,r_i} \subset V|_{D_i}).$$
2. The morphisms between two objects $(V, \mathcal{V}_1, \dots, \mathcal{V}_s)$ and $(V', \mathcal{V}'_1, \dots, \mathcal{V}'_s)$ are the morphisms of vector bundles $\phi : V \to V'$ such that, for any $1 \le i \le s$, the restriction $\phi|_{D_i}$ is compatible with the filtrations $\mathcal{V}_i$ and $\mathcal{V}'_i$, i.e., for any $1 \le j \le r_i$, $\phi|_{D_i}(V_{i,j}) \subset V'_{i,j}$.

We will say that a parabolic bundle $(V', \mathcal{V}'_1, \dots, \mathcal{V}'_s)$ is a parabolic subbundle of $(V, \mathcal{V}_1, \dots, \mathcal{V}_s)$ if and only if $V' \subset V$ is a subbundle and the inclusion map $\iota : V' \to V$ is a morphism between $(V', \mathcal{V}'_1, \dots, \mathcal{V}'_s)$ and $(V, \mathcal{V}_1, \dots, \mathcal{V}_s)$ in the category $\mathscr{P}(X, D, \underline{r})$. In this case we will write
$$(V', \mathcal{V}'_1, \dots, \mathcal{V}'_s) \subset (V, \mathcal{V}_1, \dots, \mathcal{V}_s).$$

By Theorem 5.5 for any $1 \le i \le s$ there is a manifold $\pi_i : X(D_i, r_i) \to X$ acted on by $G_i := (\mathbb{C}^\times)^{r_i}$ and a section $\sigma_i : X \to X(D_i, r_i)$ of π_i. Let us define now $X(D, \underline{r})$ to be the fibred product
$$X(D, \underline{r}) = X(D_1, r_1) \times_X X(D_2, r_2) \times_X \dots \times_X X(D_s, r_s).$$

This is smooth because the divisors D_i intersect transversely.

Denote by $\Pi : X(D,\underline{r}) \to X$ the projection. The sections σ_i induce maps $s_i : X(D_i,r_i) \to X(D,\underline{r})$. If we identify $X(D_i,r_i)$ with $X \times_X \cdots \times_X X(D_i,r_i) \times_X \cdots \times_X X$ then $s_i = (\sigma_1,\dots,\sigma_{i-1},\mathrm{Id},\sigma_{i+1},\dots,\sigma_s)$. Using the sections σ_i we also get a section $\Sigma : X \to X(D,\underline{r})$ of the projection Π. It is easy to check that for any i we have

$$\Sigma = s_i\sigma_i. \tag{8.24}$$

Consider the diagonal action of $\Gamma = G_1 \times \cdots \times G_s$ on $X(D,\underline{r})$. We define the category $\mathscr{V}_\Gamma(X(D,\underline{r}))$ as follows:

1. The objects of $\mathscr{V}_\Gamma(X(D,\underline{r}))$ are Γ-equivariant vector bundles $W \to X(D,\underline{r})$ such that for any i we have $s_i^*W \in \mathscr{V}_{G(r_i)}(X(D_i,r_i))$.
2. The morphisms between two objects W, W' are the Γ-equivariant morphisms of vector bundles.

Recall that the condition $s_i^*W \in \mathscr{V}_{G(r_i)}(X(D_i,r_i))$ translates into a certain restriction on the weights of the action of Γ (and hence is purely topological).

We now define a functor $M : \mathscr{V}_\Gamma(X(D,\underline{r})) \to \mathscr{P}(X,D,\underline{r})$. To define the action of M on objects, observe that if $W \in \mathscr{V}_\Gamma(X(D,\underline{r}))$ then $\mu(r_i)s_i^*W =: (V_i,\mathcal{V}_i) \in \mathscr{P}(X,D_i,r_i)$ is a parabolic bundle on (X,D_i). The bundle V_i is by construction the restriction $s_i^*W|_{\sigma_i(X)}$ but by (8.24) this is equal to $W|_{\Sigma(X)}$. So all the bundles $V_1,\dots,V_s$ can be canonically identified with $V := W|_{\Sigma(X)}$ and hence the filtrations $\mathcal{V}_1,\dots,\mathcal{V}_s$ are parabolic structures on V over the divisor D. We set

$$M(W) := (V,\mathcal{V}_1,\dots,\mathcal{V}_s).$$

Finally, the action of M on morphisms is given by restriction on $\Sigma(X)$.

Theorem 8.1. *For any complex manifold X, any normal crossing divisor $D \subset X$ whose irreducible components $D_1,\dots,D_s$ are smooth, and any sequence of nonzero natural numbers $\underline{r} = (r_1,\dots,r_s)$, there exists*

1. *a manifold $X(D,\underline{r})$ acted on by $\Gamma = (\mathbb{C}^\times)^{|\underline{r}|}$,*
2. *an invariant projection $\Pi : X(D,\underline{r}) \to X$ with a section $\Sigma : X \to X(D,r)$,*
3. *a full subcategory $\mathscr{V}_\Gamma(X(D,\underline{r}))$ of the category of Γ-equivariant vector bundles on $X(D,\underline{r})$, and*
4. *a functor $M : \mathscr{V}_\Gamma(X(D,\underline{r})) \to \mathscr{P}(X,D,\underline{r})$*

satisfying the following properties:

(A) Let $f : Y \to X$ be a map which is transverse to D, so that $f^{-1}D \subset Y$ is a normal crossing divisor. Then there is an induced map $f_{D,\underline{r}} : Y(f^{-1}D,\underline{r}) \to X(D,\underline{r})$ so that the following two diagrams commute:

$$\begin{tikzcd} Y(f^{-1}D,\underline{r}) \arrow[r, "f_{D,\underline{r}}"] \arrow[d] & X(D,\underline{r}) \arrow[d] \\ Y \arrow[r, "f"] & X, \end{tikzcd} \qquad \begin{tikzcd} \mathscr{V}_\Gamma(X(D,\underline{r})) \arrow[r, "f^*_{D,\underline{r}}"] \arrow[d, "M"'] & \mathscr{V}_\Gamma(Y(f^{-1}D,\underline{r})) \arrow[d, "M"] \\ \mathscr{P}(X,D,\underline{r}) \arrow[r, "f^*"] & \mathscr{P}(Y,f^{-1}D,\underline{r}); \end{tikzcd}$$

(B) the functor $M : \mathscr{V}_\Gamma(X(D,\underline{r})) \to \mathscr{P}(X,D,\underline{r})$ induces an equivalence of categories.

Proof. (A) follows from applying inductively the commutativity of diagram 4.2 and (B) is a consequence of Theorem 5.5. □

8.2. **Computing the slope.** Let us assume that X is Kaehler and compact. For any $1 \le i \le s$, let $d(i)_1, \dots, d(i)_{r_i}, t(i)_1, \dots, t(i)_{r_i}$ be the cohomology classes in $H^2(X(D_i, r_i))$ given by Lemma 6.10. Let also $p_i : X(D,\underline{r}) \to X(D_i, r_i)$ be the projection induced by $\pi_1, \dots, \pi_s$, and set, for any $1 \le j \le r_i$,

$$d_{i,j} := p_i^* d(i)_j \text{ and } t_{i,j} := p_i^* t(i)_j.$$

The following lemma follows from the definition of $X(D,\underline{r})$ and Lemma 6.10.

Lemma 8.2. *The map $\pi^* : H^2(X;\mathbb{R}) \to H^2(X(D,\underline{r});\mathbb{R})$ is injective and, identifying $H^(X;\mathbb{R})$ with its image by this map, we have*

$$H^2(X(D,\underline{r});\mathbb{R}) = H^2(X;\mathbb{R}) \oplus \bigoplus_{1\le i\le s} \bigoplus_{1\le j\le r_i} \mathbb{R}\langle d_{i,j}, t_{i,j}\rangle. \tag{8.25}$$

Let us define $\sigma = 2^{r_1} + \dots + 2^{r_s} - s$. In the following two lemmae we use the notations of Section 7 (so we use standard multiindex notation; we also denote the r_i-tuple $(d_{i,1}, \dots, d_{i,r_i})$ by d_i). The proofs of the lemmae are easy consequences of Lemmae 7.2 and 7.3.

Lemma 8.3. *Let $P_1, Q_1 \in \mathbb{Z}^{r_1}_{\ge 0}, \dots, P_s, Q_s \in \mathbb{Z}^{r_s}_{\ge 0}$. If for some $\eta \in H^*(X)$ we have $\langle \eta \prod_{1\le i\le s} d_i^{P_i} t_i^{Q_i}, [X(D,\underline{r})]\rangle \ne 0$ then (α) for any $1 \le i \le s$ we have*

$$|P_i| + |Q_i| \ge r_i \qquad \text{and} \qquad \sigma(P_i) + \sigma(Q_i) \ge 2^{r_i} - 1,$$

with equality in any of the two if and only if $P_i = (1, \dots, 1)$ and $Q_i = 0$;

(β) if $\sum_{1\le i\le s} \sigma(P_i) + \sigma(Q_i) \le \sigma$ then $\sum_{1\le i\le s} \sigma(P_i) + \sigma(Q_i) = \sigma$, and

$$\left\langle \eta \prod_{1\le i\le s} d_i^{P_i} t_i^{Q_i}, [X(D,\underline{r})] \right\rangle = \left\langle \eta \prod_{1\le i\le s} d_i^{P_i}, [X(D,\underline{r})] \right\rangle = \langle \eta, [X]\rangle.$$

Lemma 8.4. *Let $P_1, Q_1 \in \mathbb{Z}^{r_1}_{\ge 0}, \dots, P_s, Q_s \in \mathbb{Z}^{r_s}_{\ge 0}$. If for some $\eta \in H^*(X)$, $1 \le i \le s$, $1 \le j \le r_i$ we have $\langle t_{i,j} \eta \prod_{1\le u\le s} d_u^{P_u} t_u^{Q_u}, [X(D,\underline{r})]\rangle \ne 0$ then*

(α') for any $1 \le u \le s$ we have

$$|P_u| + |Q_u| \ge r_u \qquad \text{and} \qquad \sigma(P_u) + \sigma(Q_u) \ge 2^{r_u} - 1,$$

with equality in any of the two if and only if

1. *either $u \ne v$, $P_u = (1, \dots, 1)$ and $Q_u = 0$;*
2. *or $u = v$, $P_u = (0, \dots, 0, 1, \dots, 1)$ (zeroes in the first l positions, ones everywhere else) and $Q_u = (1, \dots, 1, 0, \dots, 0)$ (ones in the first l positions, zeroes everywhere else.*

(β') if $\sum_{1\leq u\leq s}\sigma(P_u)+\sigma(Q_u)\leq\sigma$ *then* $\sum_{1\leq u\leq s}\sigma(P_u)+\sigma(Q_u)=\sigma$ *and*

$$\left\langle t_{i,j}\eta\prod_{1\leq u\leq s}d_u^{P_u}t_u^{Q_u},[X(D,\underline{r})]\right\rangle=(-1)^l\langle\eta,[X]\rangle.$$

Lemma 8.5. *Let* $W\in\mathscr{V}_\Gamma(X(D,\underline{r})$*, and define* $(V,\mathcal{V}_1,\dots,\mathcal{V}_s):=M(W)\in\mathscr{P}(X,D,\underline{r})$*. Then*

$$c_1(W)=c_1(V)+\sum_{1\leq u\leq s}\sum_{1\leq l\leq r_s}R_l(\mathcal{V}_u)t(u)_l.$$

Proof. Same idea as in Lemma 7.4 (see loc. cit. for the definition of R_j). □

Fix a Kaehler class $\omega\in K(X)$ and take, for any $1\leq u\leq s$, a sequence $\underline{\lambda}_u=(\lambda_{u,1},\dots,\lambda_{u,r_u})$ of real numbers satisfying

$$1>\lambda_{u,1}>\lambda_{u,2}>\cdots>\lambda_{u,r_u}>0. \tag{8.26}$$

Let us write $\Lambda=(\underline{\lambda}_1,\dots,\underline{\lambda}_s)$. The (ω,Λ)-slope of a parabolic bundle $(V,\mathcal{V}_1,\dots,\mathcal{V}_s)\in\mathscr{P}(X,D,\underline{r})$ is by definition

$$\begin{aligned}\text{par-slope}_{\omega,\Lambda}(V,\mathcal{V}_1,\dots,\mathcal{V}_r)=\frac{1}{\operatorname{rk}V}\Bigg(&\left\langle c_1(V)\frac{\omega^{n-1}}{(n-1)!},[X]\right\rangle+\\&+\sum_{u=1}^{s}\sum_{i=1}^{r_u}R_i(\mathcal{V}_u)\lambda_{u,i}\left\langle\frac{\omega^{n-1}}{(n-1)!},[D_u]\right\rangle\Bigg).\end{aligned}$$

For any $1\leq u\leq s$, define $\beta_{u,1}=1$ and, if $1<j\leq r_u$, $\beta_{u,j}:=\lambda_{u,j}/\lambda_{u,j-1}$. For any $\epsilon\in\mathbb{R}$, let

$$\Omega(\omega,\Lambda,\epsilon):=\omega+\sum_{u=1}^{s}\sum_{j=1}^{r_u}\epsilon^{2^{j-1}}(d(u)_j-\beta_{u,j}t(u)_j). \tag{8.27}$$

Theorem 8.6. *Let* $W\in\mathscr{V}_\Gamma(X(D,\underline{r}))$*, and let* $(V,\mathcal{V}_1,\dots,\mathcal{V}_s):=M(W)\in\mathscr{P}(X,D,\underline{r})$*.*

(1) If $\epsilon>0$ *is small enough, then* $\Omega(\omega,\Lambda,\epsilon)\in K(X(D,\underline{r}))$*;*

(2) $\operatorname{slope}_{\Omega(\omega,\Lambda,\epsilon)}W$ *is a polynomial in* ϵ *of the following form:*

$$\sum_{\substack{Q=(j,j_1,\dots,j_s)\\|Q|=n}}\theta_Q(\epsilon)\langle\omega^j[D_1]^{j_1}\dots[D_s]^{j_s},[X]\rangle+\sum_{\substack{Q=(j,j_1,\dots,j_s)\\|Q|=n-1}}\theta'_Q(\epsilon)\langle c_1(V)\omega^j[D_1]^{j_1}\dots[D_s]^{j_s},[X]\rangle,$$

where both sums run over $s+1$*-tuples of nonnegative integers, and the* θ_Q*,* θ'_Q *are polynomials in* ϵ *whose coefficients only depend on* X*,* ω*,* $\underline{r}$*, and* Λ*;*

(3) we have:

$$\operatorname{slope}_{\Omega(\omega,\Lambda,\epsilon)}(W)=\epsilon^\sigma\,\text{par-slope}_{\omega,\Lambda}(V,\mathcal{V}_1,\dots,\mathcal{V}_s)+O(\epsilon^{\sigma+1}).$$

Proof. (1) follows from Lemma 6.10. The proof of the remaining points is the same as the proof of Theorem 7.5, but using Lemmae 8.3 and 8.3 instead of Lemmae 7.2 and 7.3. ☐

Remark 8.7. *It should be remarked that, when chosing the parabolic weights over an irreducible component of the divisor, one is usually allowed to set the smallest one equal to* 0*. Although we do not include this possibility in our construction (see (8.26)), it is easy to implement it. For example, if* $\lambda_{u,r_u} = 0$ *for some* $1 \leq u \leq s$*, then we set* $\beta_{u,r_u} = \epsilon$ *and define* $\Omega(\omega, \Lambda, \epsilon)$ *as in (8.27). One can then check that Theorem 8.6 remains valid.*

9. Stability

In this section we will assume that the Kaehler class $\omega \in K(X)$ is rational, i.e., there exists some $Q \in \mathbb{N}$ so that $\omega \in H^2(X; \mathbb{Z}[Q^{-1}])$. This implies that for any vector bundle $V \to X$ we have

$$\deg V \in \mathbb{Z}[(\operatorname{rk} V (n-1)! Q)^{-1}]. \tag{9.28}$$

In the sequel, whenever we talk about a subbundle F' of a vector bundle $F \to Z$, we will implicitly mean that V' is only defined over some submanifold $Z' \subset Z$, where $Z \setminus Z'$ is a subvariety of Z of codimension ≥ 2 (so, strictly speaking, V' is be a subbundle of $V|_{Z'}$). This is equivalent to saying that F' is reflexive subsheaf of the sheaf of local sections of F. We remark that the degree of such subbundles is well defined (thanks to the restriction on the codimension of $Z \setminus Z'$). Observe also that if $W \in \mathscr{V}_\Gamma(X(D, \underline{r}))$ and $(V, \mathcal{V}_1, \dots, \mathcal{V}_s) = M(W)$ then, even using this extended notion of subbundle, the functor M gives a bijection between the equivariant subbundles of W and the parabolic subbundles of $(V, \mathcal{V}_1, \dots, \mathcal{V}_s)$. This follows from the commutativity of the diagrams in Theorem 8.1 in the case $Y = X'$ and $f : X' \to X$ the inclusion.

Let us recall the notions of Mumford–Takemoto (or slope) (semi)stability for equivariant and parabolic vector bundles. We use the following standard notation: whenever we write a sentence with the word (semi)stable and the symbols $(\leq) <$ we will mean two sentences, one with the word *semistable* and the symbol $\leq$, and the other with *stable* and $<$.

Definition 9.1. *An equivariant vector bundle* $W \in \mathscr{V}_\Gamma(X(D, \underline{r}))$ *is said to be* $\Omega(\omega, \Lambda, \epsilon)$*-(semi)stable if and only if for any* $\Gamma(s)$*-equivariant subbundle* $W' \subset W$ *we have*

$$\operatorname{slope}_{\Omega(\omega,\Lambda,\epsilon)} W' (\leq) < \operatorname{slope}_{\Omega(\omega,\Lambda,\epsilon)} W.$$

We remark that if W is (semi)stable as a $\Gamma(s)$-equivariant vector bundle then it is (semi)stable as a vector bundle (i.e., the inequality between slopes holds for any subbundle of W, and not only for the equivariant ones). This follows from the existence and unicity (so in particular $\Gamma(s)$-invariance) of the Harder–Narasimhan filtration (see [GP1]).

Definition 9.2. *A parabolic bundle* $(V, \mathcal{V}_1, \dots, \mathcal{V}_s) \in \mathscr{P}(X, D, \underline{r})$ *is said to be* (ω, Λ)*-(semi)stable if and only if for any parabolic subbundle* $(V', \mathcal{V}'_1, \dots, \mathcal{V}'_s) \subset (V, \mathcal{V}_1, \dots, \mathcal{V}_s)$ *we have*

$$\text{par-slope}_{\omega,\Lambda}(V', \mathcal{V}'_1, \dots, \mathcal{V}'_s)(\leq) < \text{par-slope}_{\omega,\Lambda}(V, \mathcal{V}_1, \dots, \mathcal{V}_s).$$

Let us fix cohomology classes $c_1 \in H^2(X; \mathbb{Z})$ and $c_2 \in H^4(X; \mathbb{Z})$. We will call basic data the tuple $(X, D, \underline{r}, c_1, c_2, \omega, \Lambda)$.

Theorem 9.3. *There exists some* $\epsilon_0 > 0$*, depending only on the basic data, with the following property. Let* $W \in \mathscr{V}_\Gamma(X(D, \underline{r}))$ *and define* $(V, \mathcal{V}_1, \dots, \mathcal{V}_s) = M(W)$*. Assume that* $c_1(V) = c_1$ *and* $c_2(V) = c_2$*. Then:*

(1) If $(V, \mathcal{V}_1, \dots, \mathcal{V}_s)$ *is* (ω, Λ)*-stable, then, for any* $0 < \epsilon < \epsilon_0$*,* W *is* $\Omega(\omega, \Lambda, \epsilon)$*-stable.*

(2) If, for some $0 < \epsilon < \epsilon_0$*,* W *is* $\Omega(\omega, \Lambda, \epsilon)$*-semistable then* $(V, \mathcal{V}_1, \dots, \mathcal{V}_s)$ *is semistable.*

We will prove the theorem using statement (3) in Theorem 8.6. For that we will need to bound, uniformly over all the parabolic subbundles of $(V, \mathcal{V}_1, \dots, \mathcal{V}_s)$, the error term. This will be done in the next subsection, and the proof of the theorem will be given in Subsection 9.2.

9.1. **Bounding the error.**

9.1.1. Let E be a real $2n$-dimensional vector space, and let $J \in \operatorname{End}(E)$ satisfy $J^2 = -1$. Take on E^* the complex structure $-J^* \in \operatorname{End}(E^*)$. Let $E^*_{\mathbb{C}} = E^* \otimes_{\mathbb{R}} \mathbb{C}$. Let $a_1, \dots, a_n$ be a base of $(E^*_{\mathbb{C}})1, 0$ and let $\overline{a}_j := \overline{a_j} \in (E^*_{\mathbb{C}})^{0,1}$. Take on E an Euclidean metric which induces a Hermitian metric on $E^*_{\mathbb{C}}$ for which $a_1, \dots, a_n, \overline{a}_1, \dots, \overline{a}_n$ are orthogonal and satisfy

$$|a_j|^2 = |\overline{a}_j|^2 = 2.$$

Define $\eta = \frac{\sqrt{-1}}{2} \sum_{i=1}^n a_i \wedge \overline{a}_i \in \Lambda^{1,1} E^*_{\mathbb{C}}$. Note that the restriction of η to $E \subset E \otimes_{\mathbb{R}} \mathbb{C}$ is a real form, and so the same thing happens to any power of η. On the other hand, the restriction of $\eta^n/n! \in \Lambda^{n,n} E^*_{\mathbb{C}}$ to E coincides with the volume form induced by the chosen Euclidean metric, and we have $|\eta^n/n!| = 1$.

Lemma 9.4. *There exists a real number* $\delta_0 > 0$ *and a constant* $C > 0$ *so that, for any* $\theta \in \Lambda^{n-1,n-1} E^*_{\mathbb{C}}$ *which restricts to a real* $n-2$*-form on* E *and which satisfies* $|\eta^{n-1} - \theta| < \delta_0$*, we have:*

(1) for any $b \in (E^*_{\mathbb{C}})^{0,1}$*,* $-\sqrt{-1} b \wedge \overline{b} \wedge \theta = \beta \eta^n/n!$*, where* β *is a positive number;*

(2) for any $g \in \Lambda^2 E^*$ *we have* $g \wedge \theta = \gamma \eta^n/n!$*, where* γ *is a real number satisfying*

$$|\gamma| \leq C|g|.$$

Proof. It is clear that the map $\xi : (E^*_{\mathbb{C}})^{0,1} \ni b \mapsto -\langle \sqrt{-1} b \wedge \overline{b} \wedge \theta, \eta^n/n! \rangle$ is quadratic and takes real values. On the other hand, when $\theta = \eta^{n-1}$, the map ξ is positive definite. Since this property is preserved by slight perturbations, (1) follows. (2) is obvious. $\square$

9.1.2. Let us fix from now on an Euclidean metric on $H^{n-1,n-1}(X;\mathbb{C}) \cap H^{2n-2}(X;\mathbb{R})$, and let us denote by $B(\omega^{n-1}, r)$ the ball in $H^{n-1,n-1}(X;\mathbb{C}) \cap H^{2n-2}(X;\mathbb{R})$ centered at ω^{n-1} and with radius r.

Recall that given a metric h on a vector bundle V, there exists a unique connection A_h (the so called Chern connection) which is both compatible with h and with the complex structure of V (see [GH]). We will denote by F_h the curvature of A_h, and by $\|F_h\|_{L^2}$ the L^2 norm of F_h with respect to the metric h.

Lemma 9.5. *Let $C_0 > 0$ be any real number. There exist constants $C > 0$ and $\delta > 0$, depending only on X, ω and C_0, such that: for any vector bundle $V \to X$ admitting a metric h with $\|F_h\|_{L^2} \leq C_0$, any subbundle $V' \subset V$, and any $\Theta \in B(\omega^{n-1}, \delta)$, we have*

$$\langle c_1(V') \cap \Theta, [X]\rangle \leq C.$$

Proof. Let us begin by making some observations and definitions. There exists some $\delta_1 > 0$ and a map $\phi : B(\omega^{n-1}, \delta_1) \to \Omega^{2n-2}(X)$ such that, for any $\Theta \in B(\omega^{n-1}, \delta_1)$, $\phi(\Theta)$ is a $2n-2$-form representing Θ. Let $0 < \delta < \delta_1$ be small enough so that for any $\Theta \in B(\omega^{n-1}, \delta)$ we have $|\phi(\Theta) - \phi(\omega^{n-1})|_{C^0} < \delta_0$, where δ_0 is the number given by Lemma 9.4. Finally, let $C_0 = \sup_{\Theta \in B(\omega^{n-1},\delta)} |\phi(\Theta)|_{C^0}$.

Let $V \to X$ be a vector bundle with a metric h so that $\|F_h\|_{L^2} < C_0$. Let $V' \subset V$ be a subbundle. Using the metric h we can give a C^∞ isomorphism $V \simeq V' \oplus V''$, where $V'' = V/V'$. By means of this splitting the $\overline{\partial}$ operator of V is the following:

$$\overline{\partial}_V = \begin{pmatrix} \overline{\partial}_{V'} & \beta \\ 0 & \overline{\partial}_{V''} \end{pmatrix},$$

where $\beta \in \Omega^{0,1}(X; V''^* \otimes V')$ represents the element in $H^{0,1}(X; V''^* \otimes V')$ corresponding to the extension

$$0 \to V' \to V \to V'' \to 0.$$

A standard computation gives the following formula for the curvature F_h in terms of the splitting:

$$F_h = \begin{pmatrix} F_{h'} - \beta \wedge \beta^* & * \\ * & F_{h''} + \beta^* \wedge \beta \end{pmatrix},$$

where $F_{h'}$ (resp. $F_{h''}$) h' (resp. h'') is the restriction of h to V' (resp. V''). Let us denote by $F_h|_{V'} \in \Omega^2(X; \mathfrak{u}(V'))$ the upper left block in the matrix. By the definition of C' we have a pointwise bound

$$\operatorname{Tr} F_h|_{V'} \leq \operatorname{rk} V' |F_h| \leq \operatorname{rk} V |F_h|. \tag{9.29}$$

Let us take some $\Theta \in B(\omega^{n-1}, \delta)$, and let $\sigma = \phi(\Theta)$ be the corresponding $2n-2$-form. We then have

$$\begin{aligned} \langle c_1(V') \cap \Theta, [X]\rangle &= \int_X \frac{\sqrt{-1}}{2\pi} \operatorname{Tr} F_{h'} \wedge \sigma \\ &= \int_X \frac{\sqrt{-1}}{2\pi} \operatorname{Tr} F_h|_{V'} \wedge \sigma + \int_X \frac{\sqrt{-1}}{2\pi} \beta \wedge \beta^* \wedge \sigma \leq \operatorname{Vol}(X) \operatorname{rk} V \|F_h\|_{L^2}, \end{aligned}$$

by Cauchy–Schwarz (we have used that $\int_X \frac{\sqrt{-1}}{2\pi}\beta \wedge \beta^* \wedge \sigma < 0$, which follows from (2) in Lemma 9.4). This finishes the proof. $\square$

Lemma 9.6. *There exists a constant $C' > 0$, depending only on the basic data, such that on any (ω, Λ)-stable parabolic bundle $(V, \mathcal{V}_1, \dots, \mathcal{V}_s) \in \mathscr{P}(X, D, \underline{r})$, with $c_1(V) = c_1$ and $c_2(V) = c_2$, there is a metric h with $\|F_h\|_{L^2} < C'$.*

Proof. Let $V \to X$ be a vector bundle. Denote by $\Lambda : \Omega^2(X) \to \Omega^0(X)$ the adjoint of the map $\cdot \wedge \omega : \Omega^0(X) \to \Omega^2(X)$. For any metric h on V we have

$$\|F_h\|^2_{L^2} = \|\Lambda F_h\|^2_{L^2} - 8\pi^2 ch_2(V),$$

where $ch_2(V) = \frac{1}{2}c_1(V)^2 - c_2(V)$ (this follows from a simple computation). Hence, it suffices to find a metric h on V with $\|\Lambda F_h\|_{L^2} < C''$, where C'' depends only on the basic data.

Suppose that $(V, \mathcal{V}_1, \dots, \mathcal{V}_s)$ is (ω, Λ)-stable. It follows that there is a constant K (depending only on the basic data) such that for any subbundle $V' \subset V$ we have

$$\operatorname{slope}_\omega V' < \operatorname{slope}_\omega V + K. \tag{9.30}$$

Using the Harder–Narasimhan and Jordan–Hölder theorems (see [K]) we get a filtration

$$V_1 \subset V_2 \subset \cdots \subset V_j = V,$$

where each quotient V_{i+1}/V_i is stable and $\operatorname{slope}(V_{i+1}/V_i)$ is bounded by a function of the basic data, thanks to (9.30). By the Hitchin–Kobayashi correspondence (see [UY]) each quotient V_{i+1}/V_i admits a metric h_i with

$$\Lambda F_{h_i} = \operatorname{slope}(V_{i+1}/V_i).$$

So to get the result it suffices to prove this fact: if

$$0 \to F' \to F \to F'' \to 0 \tag{9.31}$$

is an exact sequence of vector bundles and F' (resp. F'') admits a metric h' (resp. h'') such that $\|\Lambda F_{h'}\| < K'$ and $\|\Lambda F_{h''}\| < K''$ then F admits a metric h satisfying $\|\Lambda F_h\| < K' + K''$. Now, just as in the preceding lemma, we know that F is isomorphic to the vector bundle $F' \oplus F''$ endowed with the $\overline{\partial}$-operator

$$\overline{\partial}_{F',F'',\beta} = \begin{pmatrix} \overline{\partial}_{F'} & \beta \\ 0 & \overline{\partial}_{F''} \end{pmatrix},$$

where $\beta \in \Omega^{0,1}(X; F''^* \otimes F')$ represents the element in $H^{0,1}(X; F''^* \otimes F')$ corresponding to the extension 9.31. The curvature of the Chern connection w.r.t. the metric $h' \oplus h''$ and the above $\overline{\partial}$-operator depends continuously on β, and tends to $F_{h'} \oplus F_{h''}$ as the norm of β goes to zero. But if we substitute β by $\lambda\beta$ for any $\lambda \in \mathbb{C}$ we do not change the isomorphism class of the bundle. So it suffices to take λ small enough and we are done. $\square$

Corollary 9.7. *There exists some constant $C'' > 0$, depending only on the basic data such that, for any (ω, Λ)-stable $(V, \mathcal{V}_1, \dots, \mathcal{V}_s)$ satisfying $c_1(V) = c_1$ and $c_2(V) = c_2$, any subbundle $V' \subset V$, and any $s+1$-tuple of nonnegative integers $J = (j, j_1, \dots, j_s)$ such that $|J| = n-1$, we have*

$$|\langle c_1(V')\omega^j[D_1]^{j_1} \dots [D_s]^{j_s}, [X]\rangle| \le C''(1 + |\langle c_1(V')\omega^{n-1}, [X]\rangle|).$$

Proof. Let $\epsilon > 0$ be small enough so that for any $J = (j, j_1, \dots, j_s)$ we have $\omega^{n-1} \pm \epsilon\omega^j[D_1]^{j_1} \dots [D_s]^{j_s} \in B(\omega^{n-1}, \delta)$. The preceding two lemmae imply that for any $J = (j, j_1, \dots, j_s)$

$$\pm\epsilon\langle c_1(V')\omega^j[D_1]^{j_1} \dots [D_s]^{j_s}, [X]\rangle \le C - \langle c_1(V')\omega^{n-1}, [X]\rangle,$$

from which the result follows easily. □

9.2. **Proof of Theorem 9.3.** In all this subsection, whenever we say *for small enough* ϵ, we will implicitly mean *depending only on the basic data*.

(1) Let us take an equivariant vector bundle $W \in \mathscr{V}_\Gamma(X(D, \underline{r}))$, and define $(V, \mathcal{V}_1, \dots, \mathcal{V}_s) = M(W)$. Assume that $(V, \mathcal{V}_1, \dots, \mathcal{V}_s)$ is (ω, Λ)-stable. Let $R = \operatorname{rk} V$. By (9.28), for any subbundle $V' \subset V$ we have $\deg V' \in \mathbb{Z}[(R!(n-1)!Q)^{-1}]$. This, together with the stability condition implies the existence of some $\alpha > 0$ such that for any parabolic subbundle $(V', \mathcal{V}'_1, \dots, \mathcal{V}'_s) \subset (V, \mathcal{V}_1, \dots, \mathcal{V}_s)$ we have

$$\begin{aligned}\text{par-slope}_{\omega,\Lambda}(V', \mathcal{V}'_1, \dots, \mathcal{V}'_s) + \alpha|\,\text{par-slope}_{\omega,\Lambda}(V', \mathcal{V}'_1, \dots, \mathcal{V}'_s)| \\ \le \text{par-slope}_{\omega,\Lambda}(V, \mathcal{V}_1, \dots, \mathcal{V}_s) - \alpha. \qquad (9.32)\end{aligned}$$

We want to prove that, for small enough ϵ, W is $\Omega(\omega, \Lambda, \epsilon)$-stable. In other words: letting $W' \subset W$ be a $\Gamma(s)$-equivariant subbundle, we want to check that for small enough ϵ

$$\text{slope}_{\Omega(\omega,\Lambda,\epsilon)} W' < \text{slope}_{\Omega(\omega,\Lambda,\epsilon)} W.$$

Let $(V', \mathcal{V}'_1, \dots, \mathcal{V}'_s) = M(W')$. By Theorem 8.1 $(V', \mathcal{V}'_1, \dots, \mathcal{V}'_s) \subset (V, \mathcal{V}_1, \dots, \mathcal{V}_s)$, so the inequality (9.32) holds. By (3) in Theorem 8.6 for small enough $\epsilon > 0$ we have

$$\text{slope}_{\Omega(\omega,\Lambda,\epsilon)} W > \epsilon^\sigma \left(\text{par-slope}_{\omega,\Lambda}(V, \mathcal{V}_1, \dots, \mathcal{V}_s) - \frac{\alpha}{2}\right).$$

On the other hand, by (2) and (3) in Theorem 8.6 we have

$$\begin{aligned}\text{slope}_{\Omega(\omega,\Lambda,\epsilon)} W' &= \epsilon^\sigma(\text{par-slope}_{\omega,\Lambda}(V', \mathcal{V}'_1, \dots, \mathcal{V}'_s)) \\ &\quad + \epsilon^{\sigma+1}\left(\sum_J \epsilon^{\deg(J)}\theta_J E_J + \sum_J \epsilon^{\deg(J)}\theta'_J F_J\right),\end{aligned}$$

where J denotes tuples $(j, j_1, \dots, j_s)$, $\deg(J) \ge 0$ for any J, θ_J and θ'_J are numbers which only depend on the basic data, the E_J's are of the form $\langle\omega^j[D_1]^{j_1} \dots [D_s]^{j_s}, [X]\rangle$ (so they also depend only on basic data) and where, finally, the F_J's are of the form

$$F_J = \langle c_1(V')\omega^j[D_1]^{j_1} \dots [D_s]^{j_s}, [X]\rangle.$$

By Corollary 9.7 we have bounds

$$|F_J| \le C''(1 + |\,\text{slope}_\omega V'|),$$

(C'' depends only on basic data) so, taking into account that

$$|\operatorname{slope}_\omega V'| \leq |\operatorname{par-slope}_{\omega,\Lambda}(V', \mathcal{V}'_1, \dots, \mathcal{V}'_s)| + C'''$$

for some C''' depending on the basic data, it follows that, for small enough ϵ, we have

$$\operatorname{slope}_{\Omega(\omega,\Lambda,\epsilon)} W' < \epsilon^\sigma \left(\operatorname{par-slope}_{\omega,\Lambda}(V', \mathcal{V}'_1, \dots, \mathcal{V}'_s) + \alpha|\operatorname{par-slope}_{\omega,\Lambda}(V', \mathcal{V}'_1, \dots, \mathcal{V}'_s)| + \frac{\alpha}{2}\right),$$

and we are done.

The proof of (2) follows exactly the same lines, so is left to the reader.

10. The case $\dim_{\mathbb{C}} X = 1$ and $r = 1$

Suppose that X is a compact Riemann surface (so that D is a finte set of points) and that $r = 1$. Under these hypothesis, we obtain a stronger result than in the general case. Before stating it, let us recall that we have $H^2(X_D) = H^2(X) \oplus \mathbb{R}\langle d, t\rangle$ (by Lemma 6.9) and that $td = d^2 = tp^*\alpha = 0$ (here $\alpha \in H^2(X)$ is arbitrary and $p : X_D \to X$ denotes the projection), $\langle d(p^*\alpha), [X_D]\rangle = \langle \alpha, [X]\rangle$, and $t^2 = -1$ (see Lemma 7.1).

Theorem 10.1. *Let $W \in \mathscr{V}_{\mathbb{C}^\times}(X_D)$ and let $(V, V') = \mu(W) \in \mathscr{P}(X, D)$. Let $0 < \alpha < 1$ and let $\omega \in H^2(X)$ be a Kaehler class. Define $\Omega = p^*\omega - \alpha t$. We then have*

(1) Ω is a Kaehler class of X_D,

(2) $\operatorname{slope}_\Omega(W) = \operatorname{par-slope}_{\omega,\alpha}(V, V')$,

(3) W is Ω-(semi)stable $\iff$ (V, V') is (ω, α)-parabolic (semi)stable,

(4) fix some $d \in H^2(X; \mathbb{Z})$, some nonzero $r \geq r' \in \mathbb{N}$; let $\mathscr{M}_{d,r,r',\alpha}(X)$ be the moduli space of (ω, α)-stable parabolic vector bundles (V, V') over (X, D) satisfying $c_1(V) = d$, $\operatorname{rk} V = r$, $\operatorname{rk} V' = r'$; let $\mathscr{M}_\Omega(X_D)$ be the moduli space of vector bundles $W \to X_D$ with Chern character $\operatorname{ch} W = p^*(r + d) + tr'$; then $\mathbb{C}^\times$ acts algebraically on $\mathscr{M}_\Omega(X_D)$ and $\mathscr{M}_{d,r,r',\alpha}$ can be identified with some of the connected components of the fixed point set of this action.

Proof. (1) follows from Theorem 6.8; (2) can be checked by computing both sides, using the relations between d and t which we recalled above; (3) is a consequence of (2); finally, (4) follows from (2), the ideas in Section 5.1 of [GP2], and the fact that Theorem 4.2 works also for families of vector bundles — this is true because the only technical tool which was used in the proof was Riemann's extension theorem, which is certainly true for pairs $(X \times S, D \times S)$ for any scheme S of finite type. □

We observe that not only Theorem 4.2 but its most general version, Theorem 8.1, is valid for families of vector bundles (by exactly the same argument as before). Unfortunately, this does not allow to identify the moduli spaces of equivariant and parabolic vector bundles, since the best thing we could do in general was to identify the notion of (ω, Λ)-parabolic stability to that of $\Omega(\omega, \Lambda, \epsilon)$-stability as $\epsilon \to 0$. If we knew that the notion of $\Omega(\omega, \Lambda, \epsilon)$-stability for vector bundles over $X(D, \underline{r})$ does not change when $\epsilon > 0$ is small enough, then we could identify the moduli spaces. But this does not seem to be the case in general (see for example [Q, S]).

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